Choosing Your Own Luck: Strategic Risk Taking in Contests

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Introduction

- ***Contests*** are incentive schemes based on ordinal performance comparisons
- Used extensively in organizations and beyond: sales contests, competition for promotion, political competition, R&D, etc.
- Contest models: agents exert costly *effort* (or make costly investments) to secure a prize
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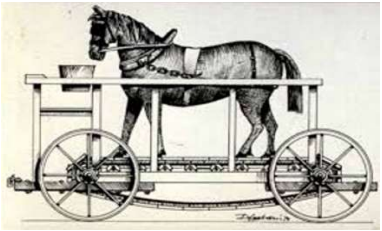
Contest Design where

Agents simultaneously, flexibly choose

- effort
- risk

technology choice $=$ ***Effort*** $+$ ***Risk***

Rainhill Trials: October 1929



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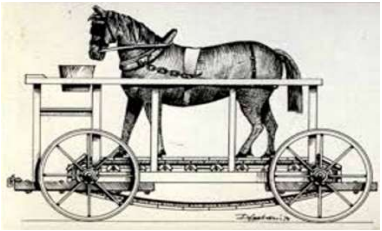

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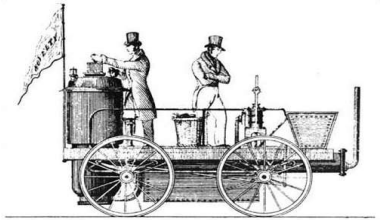

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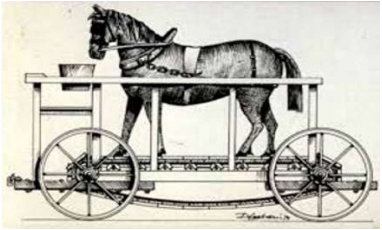

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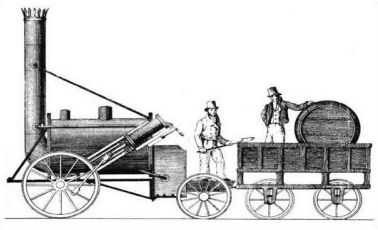

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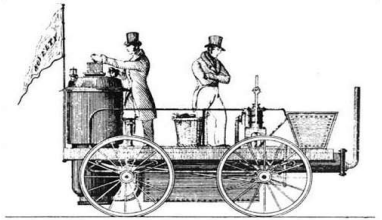

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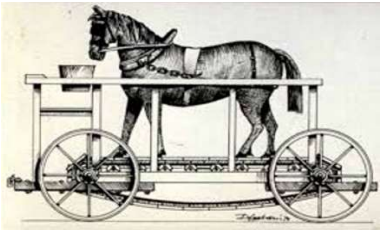

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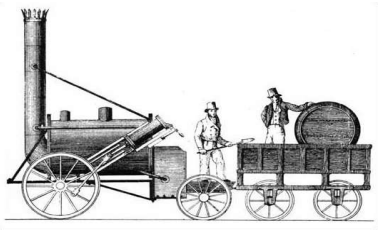

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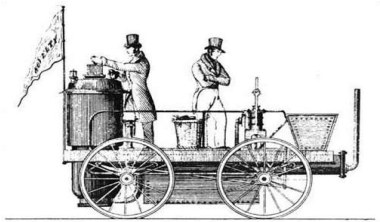

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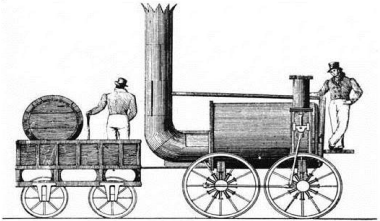

Sans Pareil

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Contest Design Problem

What is optimal $\mathbf{v}$ that maximises $\mathbf{E}[X_i]$?

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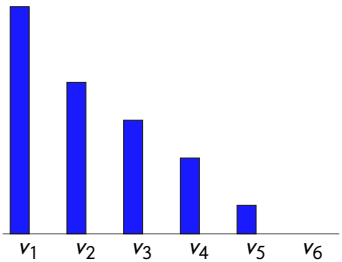

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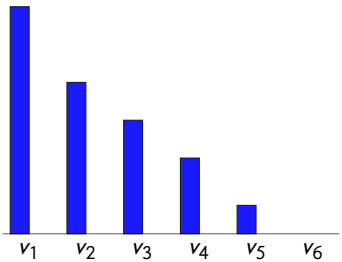

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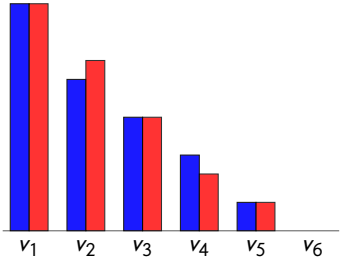

Common theme in the literature: Prize sharing

- Cost convexity Fang, Noe, and Strack (2020), Moldovanu and Sela (2001)
- Heavy-tailed noise Drugov and Ryvkin (2020)
- Other factors
  - Risk aversion
  - Inequality aversion
- FNS + MS: WTA optimal for concave c, … details of costs matter
- This paper: ***strategic risk taking*** pushes optimal prizes towards ***less prize sharing***
  - Not sensitive to cost function details

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Symmetric Equilibrium (F^*, G^*): player i solves

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- G^* is equilibrium in risk-taking contest given F^*
  - Myerson (1993): F^* is degenerate, exogenous
  - Here, F^* ***endogenous***

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- *If* $\Phi_{\mathbf{v}}(G^*(y))$ *is strictly concave locally, then* $F^* = G^*$
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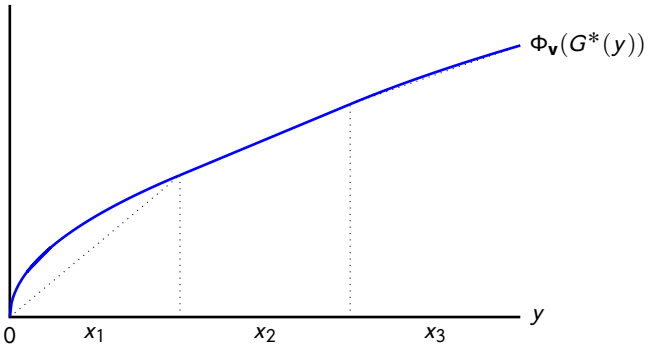

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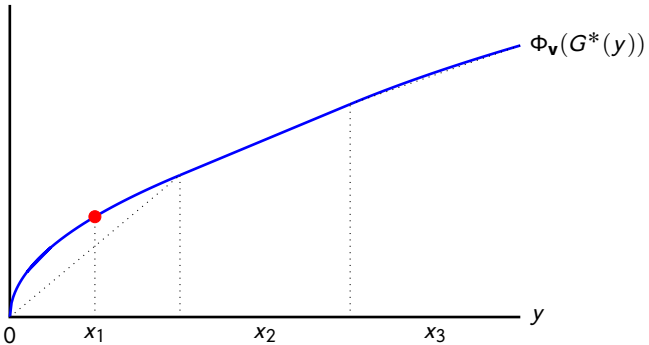

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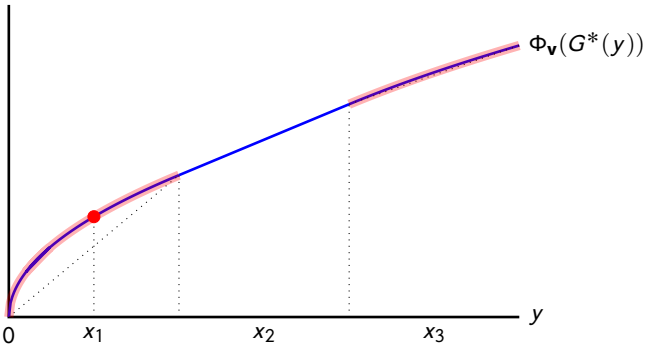

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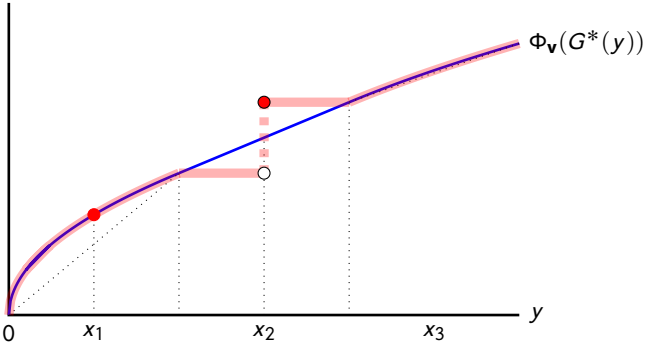

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Dworczak and Martini (2019)

$$\max_{F \in \Delta(\mathbb{R}_+)} \int u(x)\, \mathrm{d}F(x) \quad \text{s.t. } F \in \mathrm{MPC}(G^*)$$

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$$\min \int \phi \, \mathrm{d}G^* \quad \text{s.t. } \phi \text{ convex}, \phi \geqslant u$$

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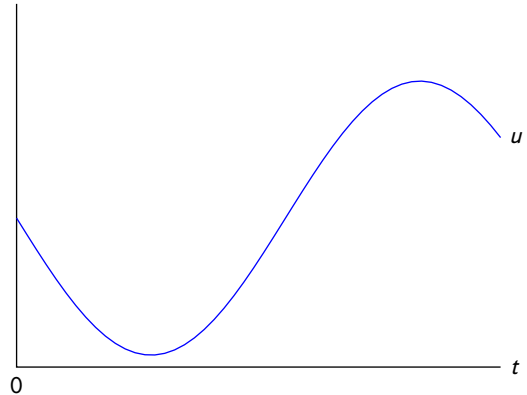

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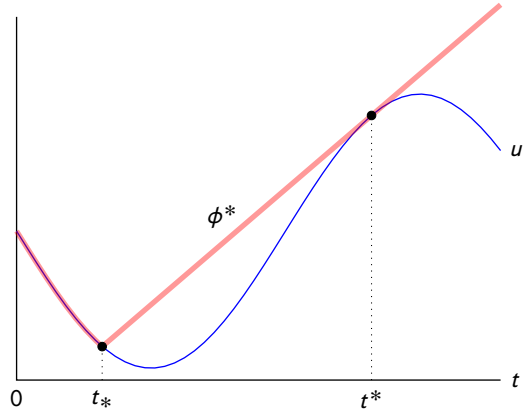
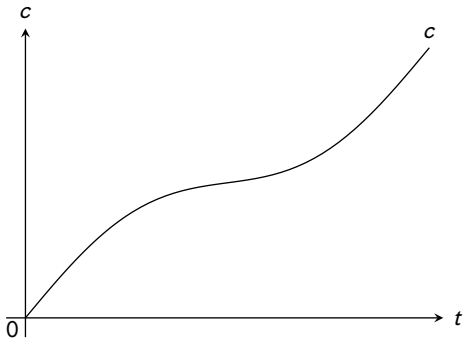


Applying DM to Our Problem



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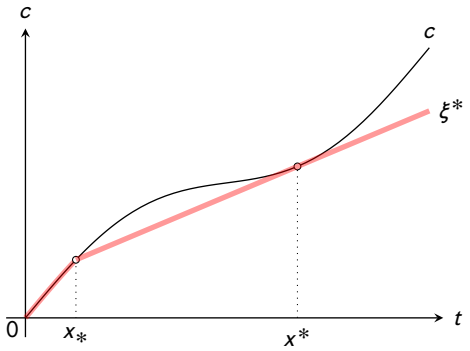

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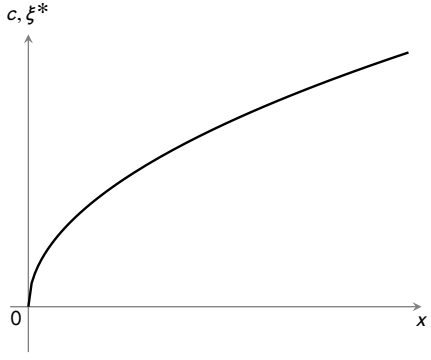
Proposition

The equilibrium output distribution G^* satisfies

$$\Phi_{\mathbf{v}}(G^*(y)) = \min\{\tilde{\xi}^*(y), v_1\}\ \forall y \qquad \text{if and only if} \qquad G^*(y) = \Gamma_{\mathbf{v}}(\tilde{\xi}^*(y))$$

where $\tilde{\xi}^*(y) = \xi^*(y) - \xi^*(0)$ and $\Gamma_{\mathbf{v}}(t) = \Phi_{\mathbf{v}}^{-1}(t)1(t \leqslant v_1) + 1(t > v_1)$

Equilibrium with Concave Costs



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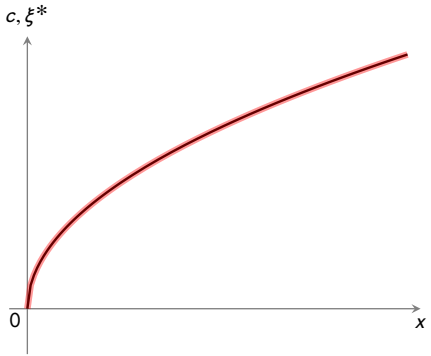
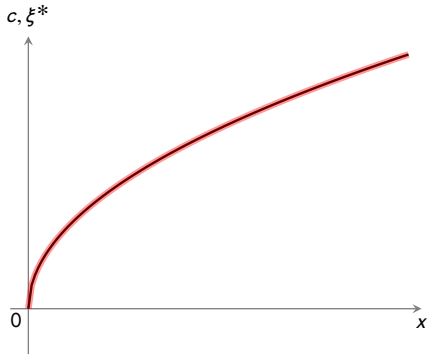


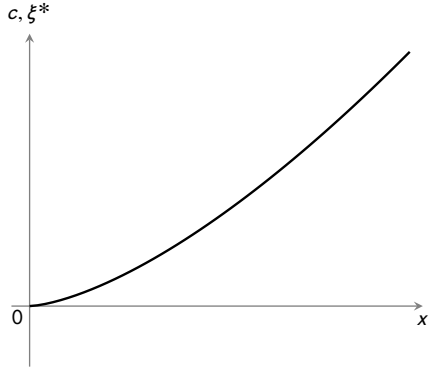
- If c is concave then $\xi^* = c \quad \Longrightarrow \quad G^*(y) = \Gamma_{\mathbf{v}}(c(y))$ (as without risk taking)

Equilibrium with Concave Costs



- If c is concave then $\xi^* = c \quad \Longrightarrow \quad G^*(y) = \Gamma_{\mathbf{v}}(c(y))$ (as without risk taking)
- Because $\Phi(G^*; \mathbf{v}) = \xi^* = c$ is strictly concave, $F^* = G^*$

Equilibrium with Convex Costs



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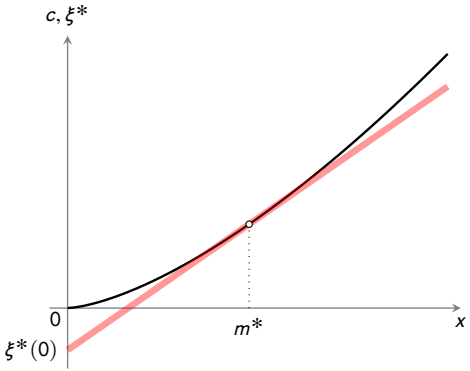
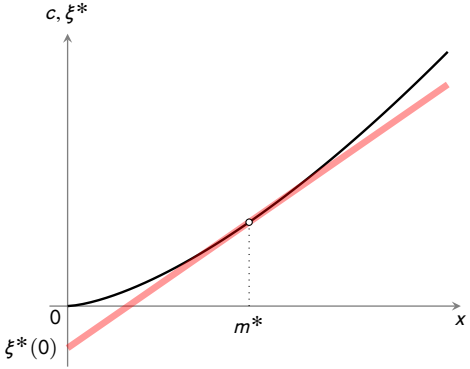


- If c is convex then ξ^* is affine and tangent to c at m^*

Equilibrium with Convex Costs



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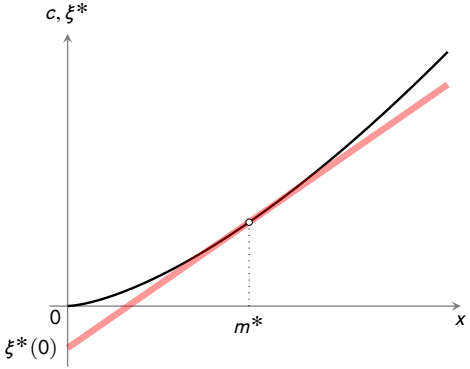

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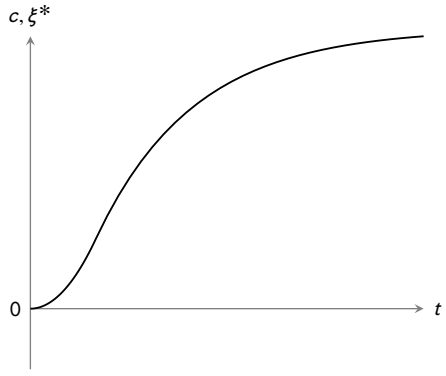
- Then, G^* is such that $\Phi(G^*(y); \mathbf{v}) = \tilde{\xi}^*(y) = \xi^*(y) - \xi^*(0) = c'(m^*)y$
- And F^* is degenerate at m^*

Equilibrium with Convex-Concave Costs



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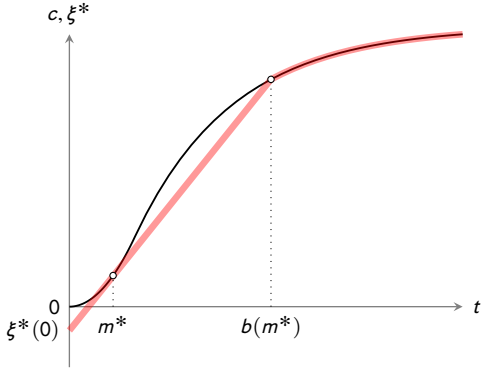
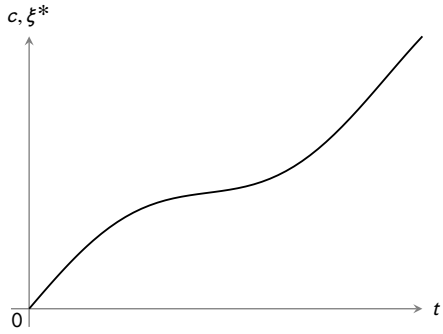


- If c is convex-concave then ξ^* is affine-concave
- F^* has mass at m^* and coincides with G^* above $b(m^*)$

Equilibrium with Concave-Convex Costs



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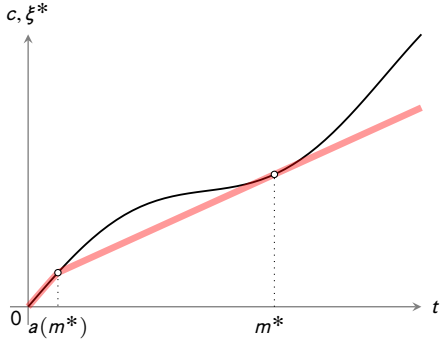

- If c is concave-convex then ξ^* is concave-affine
- $F^* = G^*$ below $a(m^*)$ and then mass at m^*

Contest Performance

- ***Question:*** What happens to $\mathbf{E}[X^*] = \mathbf{E}[Y^*]$ as $\mathbf{v}$ becomes more unequal; in particular, How does $\mathbf{v}^{\text{WTA}}$ compare to other contests?

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Proposition

If c is concave then ***No Risk Taking***, ie, $X^* = Y^*$ and $\varepsilon = 0$. Therefore, the existing results continue to apply.
If c is convex then ***Max Risk Taking***, ie, $X^* =$ degenerate, $Y^* = X^* + \varepsilon$. Virtual cost is concave, independent of $\mathbf{v}$, so existing results apply

- More inequality raises the equilibrium distribution $X^* = Y^*$ in increasing convex order
- WTA dominates any other contest in increasing convex order (so maximizes expected effort/output).

Contest Performance: General Look

- Recall equilibrium output distribution G^* given by

$$\Phi_{\mathbf{v}}(G^*(y)) = \min\{\tilde{\xi}^*(y), v_1\} \qquad \Longleftrightarrow \qquad G^*(y;\mathbf{v}) = \Gamma_{\mathbf{v}}\left(\tilde{\xi}^*(y;\mathbf{v})\right)$$

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> ***Lemma:*** Because ξ^* is always concave, the ***prize effect*** necessarily raises the distribution in ***increasing convex order***

Concave or Convex Costs

- ***c is concave:*** then $\tilde{\xi}^* = c$ independent of **v**

Concave or Convex Costs

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Suppose c is convex. Then, for any contest $\mathbf{v} \in \mathbb{V}$, there exists a unique symmetric equilibrium, where

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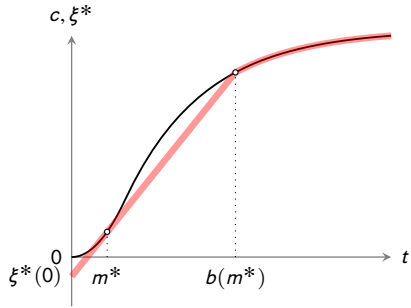
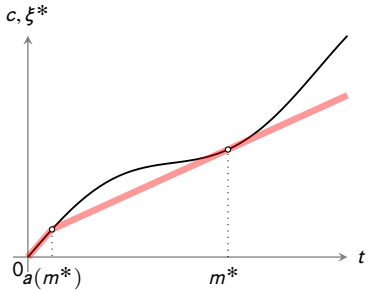
- G^* rises in ***convex order*** if $\mathbf{v}$ becomes more unequal.

Concave-Convex or Convex-Concave Costs





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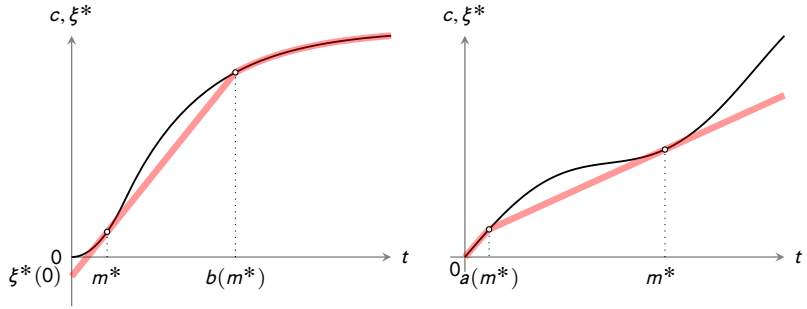

- Why? Because, unlike other studies (FNS, …), our result is unambiguous: WTA maximizes expected effort/output. ***Does it hold even beyond concave or convex costs?***

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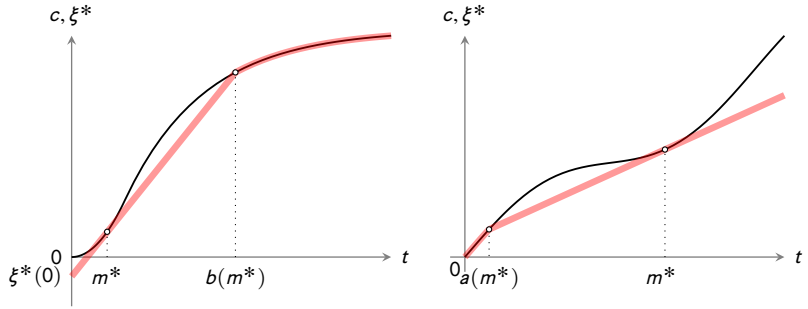

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- Convex costs mainly for ***tractability***, but …

Concave-Convex Costs: Textbook Example (Besanko and Breautigam)

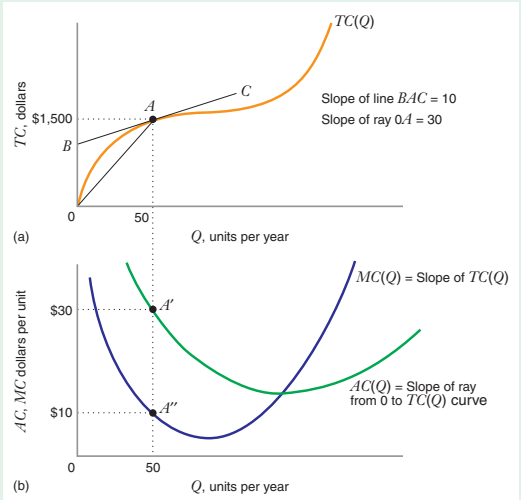

FIGURE 8.7 Deriving Long-Run Average and Marginal Cost Curves from the Long-Run Total Cost Curve
Panel (a) shows the firm's long-run total cost curve $TC(Q)$. Panel (b) shows the long-run average cost curve $AC(Q)$ and the long-run marginal cost curve $MC(Q)$, both derived from $TC(Q)$. At point A in panel (a), when output is 50 units per year, average cost = slope of ray $0A$ = \$30 per unit; marginal cost = slope of line BAC = \$10 per unit. In panel (b), points A' and A'' correspond to point A in panel (a), illustrating the relationship between the long-run total, average, and marginal cost curves.

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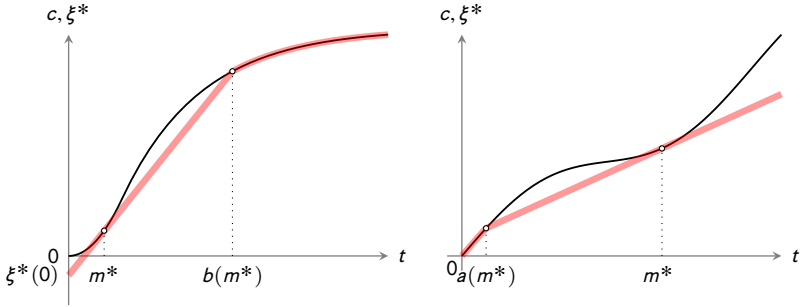

- For these cost functions, $\tilde{\xi}^*$ ***does depend*** on **v**

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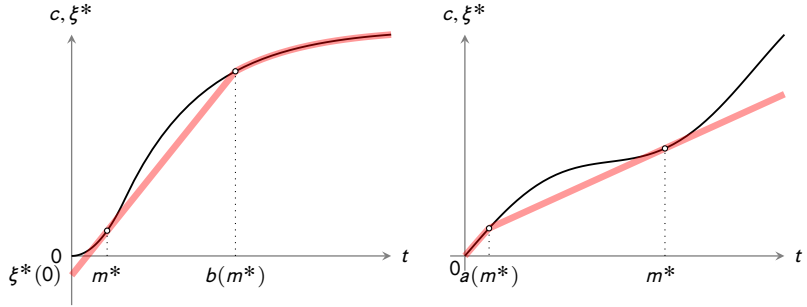

- For these cost functions, $\tilde{\xi}^*$ ***does depend*** on $\mathbf{v}$
- ***Challenge:*** uncover the direction of the virtual cost effect

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Convex-Concave Costs: Comparative Statics

Proposition

If ***c is convex-concave*** then the unique equilibrium output in the WTA contest dominates the largest equilibrium output from any other contest in ***increasing convex order***

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- Focus on that m^*

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Concave-Convex Costs: Comparative Statics

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If ***c is concave-convex***, then

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Proposition

If ***c is concave-convex***, then

- the expected virtual cost effect $= 0$
- $\mathbf{E}[Y^*(\mathbf{w})] \geqslant \mathbf{E}[Y^*(\mathbf{v})]$ whenever $\mathbf{w}$ is more unequal than $\mathbf{v}$

Summary: The Optimality of WTA contests

Consider any $\mathbf{v} \in \mathbb{V}$

- ***c is concave or convex-concave*** $\implies Y^*(\mathbf{v}^{\text{WTA}}) \succcurlyeq_{\text{icx}} Y^*(\mathbf{v})$

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  - If c is concave, convex, or convex-concave then $\mathbf{v}^{\text{WTA}}$ maximizes the (expected) highest output.

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- Maximizing highest output
  - If c is concave, convex, or convex-concave then $\mathbf{v}^{\text{WTA}}$ maximizes the (expected) highest output.
  - If c is concave-convex then the same result is likely to continue to hold, but the result cannot be generally established because the virtual cost effect works in the opposite direction from the prize effect.