Supplementary Appendix To "Dynamic Financial Contracting with Persistent Private Information"*

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In this document, we first apply the typical techniques using Bellman operator to show various properties of the firm's surplus function. Then we collect miscellaneous facts about paths of Markov chains that are useful to prove the long-run properties of the firm.

1. Bellman Operator

In this section, we will define the Bellman operator corresponding to maximizing the firm's surplus and show that the resulting value function satisfies various properties. Let $C(V \times S)$ be the space of continuous functions on the domain $V \times S$ and let $\mathscr{F} := \{P \in C(V \times S) : 0 \leq P(\mathbf{v}, s) \leq \bar{Q}(s)\}$ be endowed with the 'sup' metric, where $\bar{Q}(b)$ and $\bar{Q}(g)$ are the first best surplus specified in Section **??** of the text. It is easy to see that $\mathscr{F}$ so defined is a complete metric space. Define the Bellman operator $T : \mathscr{F} \to \mathscr{F}$ as:

[P1] $$(TP)(\mathbf{v}, s) = \max_{k, m_i, \mathbf{w}_i} \big(-k + p_s[R(k) + \delta P(\mathbf{w}_g, g)] + (1 - p_s)\delta P(\mathbf{w}_b, b) \big)$$

where $\mathbf{w}_i \in V$ and $(k, m_i, \mathbf{w}_i)$ satisfies:

[PK$_b$] $$v_b = -m_b + \delta\, \mathbb{E}^b[\mathbf{w}_b]$$

[PK$_g$] $$v_g = R(k) - m_g + \delta\, \mathbb{E}^g[\mathbf{w}_g]$$

[IC] $$v_g \geq R(k) - m_b + \delta\, \mathbb{E}^g[\mathbf{w}_b]$$

[LL] $$m_g \leq R(k), \quad m_b \leq 0$$

(*) The most recent version of the paper and the Supplementary Appendices are available here.

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Using [PK$_b$] and [PK$_g$], we can rewrite [IC] as:

[IC*] $$v_g - v_b \geq R(k) + \delta\Delta(w_{bg} - w_{bb})$$

It is standard to show that $T : \mathcal{F} \to \mathcal{F}$ is well defined. In what follows, we shall show that T maps certain subsets of $\mathcal{F}$ to themselves. We begin by showing that T maps concave functions to concave functions. To proceed, let $\mathcal{F}_1 := \{P \in \mathcal{F} : P(\cdot, s) \text{ is concave for all } s \in S\}$.

Lemma 1.1. If $P \in \mathcal{F}_1$, then $TP \in \mathcal{F}_1$.

Proof. Let $P \in \mathcal{F}_1$. For any $k \geq 0$, let $R(k) = r$, so that $c(r) = k$, where $c(r) := R^{-1}(r)$. Because R is increasing, c is well defined. The concavity of R implies that c is convex. Thus, we can let the choice variables be $(r, m_i, \mathbf{w}_i)$, so that the objective becomes

$$\begin{aligned}(TP)(\mathbf{v}, s) = \max_{r, m_i, \mathbf{w}_i} & \big(-c(r) + p_s[r + \delta P(\mathbf{w}_g, g)] + (1 - p_s)\delta P(\mathbf{w}_b, b)\big) \\ \text{s.t.} \quad & (r, m_i, \mathbf{w}_i) \in \Gamma(\mathbf{v}, s)\end{aligned}$$

It is easy to see that with this transformation, $\Gamma(\mathbf{v}, s)$ is the intersection of finitely many affine sets, and so is convex. Moreover, the objective, $-c(r)+p_s[r+\delta P(\mathbf{w}_g, g)]+(1 - p_s)\delta P(\mathbf{w}_b, b)$ is concave in $(r, \mathbf{w}_i)$. Standard arguments now imply $TP(\mathbf{v}, s) \in \mathcal{F}_1$. □

We show in the following result that the firm becomes unconstrained (firm surpluses reach first best) on the sets E'_s where

$$E'_b := \left\{\mathbf{v} \geq \bar{\mathbf{v}}_b : v_g - v_b \geq \frac{R(\bar{k}_b)}{(1 - \delta\Delta)}\right\}$$
$$E'_g := \left\{\mathbf{v} \geq \bar{\mathbf{v}}_g : v_g - v_b \geq R(\bar{k}_g) + \delta\Delta \max\left[\frac{\delta\bar{v}_{bg} - v_b}{\delta(1 - p_b)}, \frac{R(\bar{k}_b)}{1 - \delta\Delta}\right]\right\}$$

The vectors $\bar{\mathbf{v}}_b$ and $\bar{\mathbf{v}}_g$ are explicitly expressed by the parameters p_b, p_g, δ in Section **??**. In particular, we can verify that $\bar{\mathbf{v}}_s \in E'_s$ and

[1.1] $$\bar{v}_{sb} = \delta\,\mathbb{E}^b[\bar{\mathbf{v}}_b]$$

[1.2] $$\bar{v}_{sg} \geq \delta\,\mathbb{E}^g[\bar{\mathbf{v}}_g]$$

Given these properties, we now show that T maps the space of functions that achieve $\bar{Q}(s)$ on the sets E'_s to itself. Let $\mathcal{F}_2 := \{P \in \mathcal{F} : P(\mathbf{v}, s) = \bar{Q}(s) \text{ for all } \mathbf{v} \in E'_s, s \in S\}$.

Lemma 1.2. If $P \in \mathcal{F}_2$, then $TP \in \mathcal{F}_2$.[1]

Proof. Suppose $P \in \mathcal{F}_2$. We will show that for any $\mathbf{v} \in E'_s$, there exists a policy such that $TP(\mathbf{v}, s) = \bar{Q}(s)$.

Let us first consider the case where $\mathbf{v} \in E'_b$, so that $s = b$. For such a $\mathbf{v}$, consider the policy

- $k = \bar{k}_b$, $\mathbf{w}_g = \bar{\mathbf{v}}_g$
- $w_{bg} = \bar{v}_{bg}$, $w_{bb} = \bar{v}_{bg} - \min\{v_g - v_b, \bar{v}_{bg} - \bar{v}_{bb}\}$
- $m_g = R(\bar{k}_b) + \delta\,\mathbb{E}^g[\bar{\mathbf{v}}_g] - v_g$
- $m_b = \delta\,\mathbb{E}^b[\mathbf{w}_b] - v_b$

We shall first show this policy satisfies all the constraints in [P1]. The constraints [PK_b] and [PK_g] follow from the definitions of m_b and m_g. Because $\mathbf{v} \in E'_b$, we have

$$\begin{aligned} v_g - v_b &\geq R(\bar{k}_b) + \delta\Delta(v_g - v_b) \\ &\geq R(\bar{k}_b) + \delta\Delta \min\{v_g - v_b, \bar{v}_{bg} - \bar{v}_{bb}\} \\ &= R(\bar{k}_b) + \delta\Delta(w_{bg} - w_{bb}) \end{aligned}$$

The first line is from the definition of E'_b while the second is just arithmetic. So the constructed policy satisfies [IC*]. By the definition of w_{bb}, we either have $w_{bb} = \bar{v}_{bb}$ or $w_{bb} = v_b - (v_g - \bar{v}_{bg}) \leq v_b$; the definition of w_{bb} also implies that in this latter case, $w_{bb} = v_b - (v_g - \bar{v}_{bg}) \geq v_b - (\bar{v}_{bg} - \bar{v}_{bb}) = \bar{v}_{bb}$.

In sum, we obtain $\bar{v}_{bb} \leq w_{bb} \leq v_b$, so from [1.1] it follows that

$$\begin{aligned} v_b \geq w_{bb} &\geq \delta[p_b \bar{v}_{bg} + (1 - p_b) w_{bb}] \\ &= \delta\,\mathbb{E}^b[\mathbf{w}_b] \end{aligned}$$

which means the constructed transfer $m_b \leq 0$, ie, [LL] for b is satisfied. Moreover, the constructed transfer m_g satisfies

$$m_g \leq R(\bar{k}_b) + \delta\,\mathbb{E}^g[\bar{\mathbf{v}}_g] - \bar{v}_{bg} \leq R(\bar{k}_b)$$

The first inequality is from $\bar{v}_{bg} \leq v_g$, and the second is by [1.2]. Moreover, $m_g \leq R(\bar{k}_b)$ implies [LL] for g is satisfied.

By definition, we have $w_{bg} - w_{bb} = \min\{v_g - v_b, \bar{v}_{bg} - \bar{v}_{bb}\} \geq \frac{R(\bar{k}_b)}{1-\delta\Delta}$, where the inequality is because $\mathbf{v}, \bar{\mathbf{v}}_b \in E'_b$. Therefore, our choice of $\mathbf{w}_b \in E'_b$, which in conjunction with the assumption that $P \in \mathcal{F}_2$ implies $P(\mathbf{w}_b, b) = P(\bar{\mathbf{v}}_b, b)$.

As all the constraints of [P1] are satisfied, we have

$$\begin{aligned} \bar{Q}(b) &\geq TP(\mathbf{v}, b) \\ &\geq -\bar{k}_b + p_b[R(\bar{k}_b) + \delta P(\bar{\mathbf{v}}_g, g)] + (1 - p_b)\delta P(\mathbf{w}_b, b) \\ &= \bar{Q}(b) \end{aligned}$$

(1) As is clear from Proposition 4.2 in the text, $E'_s = E_s$ and is therefore the largest set with this property.

where we have used the definition of $\bar{Q}$, and the facts that $\bar{\mathbf{v}}_g \in E'_g$, $P \in \mathcal{F}_2$, and $P(\mathbf{w}_b, b) = P(\bar{\mathbf{v}}_b, b)$ to obtain the last equality. Therefore $TP(\mathbf{v}, b) \in \mathcal{F}_2$.

Now consider the case where $\mathbf{v} \in E'_g$ and $s = g$. For such a $\mathbf{v}$, consider the policy

- $k = \bar{k}_g$, $\mathbf{w}_g = \bar{\mathbf{v}}_g$
- $w_{bg} = \bar{v}_{bg}$, $w_{bb} = \bar{v}_{bg} - \max\left[\frac{\delta \bar{v}_{bg} - v_b}{\delta(1-p_b)}, \frac{R(\bar{k}_b)}{1-\delta\Delta}\right]$
- $m_g = R(\bar{k}_g) + \delta\, \mathbb{E}^g[\bar{\mathbf{v}}_g] - v_g$
- $m_b = \delta\, \mathbb{E}^b[\mathbf{w}_b] - v_b$

The constraints [PK_b], [PK_g] are satisfied by construction. Because $\mathbf{v} \in E'_g$, we find

$$
\begin{aligned}
v_g - v_b &\geq R(\bar{k}_g) + \delta\Delta \max\left[\frac{\delta \bar{v}_{bg} - v_b}{\delta(1-p_b)}, \frac{R(\bar{k}_b)}{1-\delta\Delta}\right] \\
&= R(\bar{k}_g) + \delta\Delta(w_{bg} - w_{bb})
\end{aligned}
$$

which means [IC*] is satisfied. The constructed transfer m_g satisfies:

$$
m_g \leq R(\bar{k}_g) + \delta\, \mathbb{E}^g[\bar{\mathbf{v}}_g] - \bar{v}_{gg} \leq R(\bar{k}_g)
$$

where the first inequality is because $\bar{v}_{gg} \leq v_g$, and the second follows from [1.2]. So [LL] for g is satisfied. The constructed transfer m_b satisfies

$$
\begin{aligned}
m_b &= \delta\, \mathbb{E}^b[\mathbf{w}_b] - v_b \\
&= \delta \bar{v}_{bg} - \delta(1-p_b) \max\left[\frac{\delta \bar{v}_{bg} - v_b}{\delta(1-p_b)}, \frac{R(\bar{k}_b)}{1-\delta\Delta}\right] - v_b \\
&\leq 0
\end{aligned}
$$

so that [LL] for b is satisfied. Therefore, all the constraints of [P1] are satisfied, and the policy is feasible.

Notice also that

$$
\begin{aligned}
w_{bb} &\geq \bar{v}_{bg} - \max\left[\frac{\delta \bar{v}_{bg} - \bar{v}_{bb}}{\delta(1-p_b)}, \frac{R(\bar{k}_b)}{1-\delta\Delta}\right] \\
&\geq \bar{v}_{bg} - (\bar{v}_{bg} - \bar{v}_{bb}) = \bar{v}_{bb}
\end{aligned}
$$

The first line is from $\bar{v}_{bb} \leq v_b$, while the second line follows because $\frac{\delta \bar{v}_{bg} - \bar{v}_{bb}}{\delta(1-p_b)} = \bar{v}_{bg} - \bar{v}_{bb}$ (by [1.1]) and $\frac{R(\bar{k}_b)}{1-\delta\Delta} \leq \bar{v}_{bg} - \bar{v}_{bb}$ (from the definition of E'_g). By construction, $w_{bg} - w_{bb} \geq \frac{R(\bar{k}_b)}{1-\delta\Delta}$, so it follows that $\mathbf{w}_b \in E'_b$. The assumption $P \in \mathcal{F}_2$ then implies $P(\mathbf{w}_b, b) = P(\bar{\mathbf{v}}_b, b)$. Putting this all together, we obtain

$$
\begin{aligned}
\bar{Q}(g) &\geq TP(\mathbf{v}, g) \\
&\geq -\bar{k}_g + p_g[R(\bar{k}_g) + \delta P(\bar{\mathbf{v}}_g, g)] + (1-p_g)\delta P(\mathbf{w}_b, b) \\
&= \bar{Q}(g)
\end{aligned}
$$

where we have used the definition of $\bar{Q}(g)$ and the facts that $\bar{\mathbf{v}}_s \in E'_s$, $P \in \mathcal{F}_2$, and $P(\mathbf{w}_s, s) = P(\bar{\mathbf{v}}_s, s)$ for $s = b, g$. Therefore $TP(\mathbf{v}, g) \in \mathcal{F}_2$, as desired. □

We now show that T preserves functions that are locally decreasing in v_b at (v, v) for $v > 0$. More precisely, let

$$\mathcal{F}_3 = \{P(\mathbf{v}, s) \in \mathcal{F} : P((v, v), s) \text{ is local minimum of } P((v_b, v), s)\ \forall v > 0\}$$

Lemma 1.3. If $P(\mathbf{v}, s) \in \mathcal{F}_3$, then $TP(\mathbf{v}, s) \in \mathcal{F}_3$.

Proof. Suppose $P(\mathbf{v}, s) \in \mathcal{F}_3$ and pick any $v > 0$. Let $(k, m_b, m_g, \mathbf{w}_b, \mathbf{w}_g)$ be the optimal policy at $((v, v), s)$. From [IC*], we must have $k = 0$ and $w_{bg} = w_{bb}$. From [PK$_b$], we then obtain $w_{bg} = w_{bb} = (v + m_b)/\delta$. Let $v' = v - \frac{1-p_b}{1-p_g} R(\varepsilon)$ for any arbitrary small $\varepsilon > 0$, and $k' = R(\varepsilon)$, $w'_{bb} = w_{bb} - \frac{R(\varepsilon)}{\delta(1-p_g)}$.

It is easy to verify that $(k', m_b, m_g, (w'_{bb}, w_{bg}), \mathbf{w}_g) \in \Gamma((v', v), s)$, ie, it is a feasibly policy, so that we have

$$\begin{aligned} &TP((v', v), s) - TP((v, v), s) \\ &\quad \geq -\varepsilon + p_s R(\varepsilon) + (1 - p_s)\delta \left[P\left((w'_{bb}, w_{bg}), b\right) - P((w_{bb}, w_{bg}), b)\right] \\ &\quad \geq 0 \end{aligned}$$

The last inequality is implied by the fact that $P \in \mathcal{F}_3$ and the assumption that $R'(0) = \infty$, which implies $p_s R(\varepsilon) > \varepsilon$ for $\varepsilon > 0$ sufficiently small. (In particular, all we require is that ε be such that $p_s R(\varepsilon) \geq \varepsilon$), which defines a large interval containing the efficient levels of investment in both states.) This implies that $TP((v_b, v), s)$ is locally minimised at $v_b = v$, ie, $TP(\mathbf{v}, s) \in \mathcal{F}_3$. □

We now show that T preserves functions that generate higher value when $s = g$ than when $s = b$. In particular, let $\mathcal{F}_4 := \{P(\mathbf{v}, s) \in \mathcal{F} : P(\mathbf{v}, g) \geq P(\mathbf{v}, b)\}$.

Lemma 1.4. If $P(\mathbf{v}, s) \in \mathcal{F}_4$, then $TP(\mathbf{v}, s) \in \mathcal{F}_4$.

Proof. Suppose $P(\mathbf{v}, s) \in \mathcal{F}_4$. Let $(k, m_b, m_g, \mathbf{w}_b, \mathbf{w}_g)$ be the optimal policy at $(\mathbf{v}, b)$. By [IC], we know

$$v_g \geq R(k) - m_b + \delta\, \mathbb{E}^g[\mathbf{w}_b] \geq \delta\, \mathbb{E}^g[\mathbf{w}_b]$$

The second inequality is from [LL] which requires $m_b \leq 0$. Let $m'_g = \delta\, \mathbb{E}^g[\mathbf{w}_b] - v_g + R(k)$ which implies $m'_g \leq R(k)$. Thus, the policy $(k, m_b, m'_g, \mathbf{w}_b, \mathbf{w}_b)$ satisfies [PK$_g$] and [LL] for g. As other constraints do not change, we know that $(k, m_b, m'_g, \mathbf{w}_b, \mathbf{w}_b) \in \Gamma(\mathbf{v}, b)$, which implies

$$\begin{aligned} TP(\mathbf{v}, b) &= -k + p_b R(k) + \delta p_b [P(\mathbf{w}_g, g) - P(\mathbf{w}_b, b)] + \delta P(\mathbf{w}_b, b) \\ &\geq -k + p_b R(k) + \delta p_b [P(\mathbf{w}_b, g) - P(\mathbf{w}_b, b)] + \delta P(\mathbf{w}_b, b) \end{aligned} \qquad [1.3]$$

where the inequality reflects the choice of a (possibly) suboptimal policy. From [1.3] we then get

$$[1.4] \qquad P(\mathbf{w}_g, g) - P(\mathbf{w}_b, b) \geq P(\mathbf{w}_b, g) - P(\mathbf{w}_b, b) \geq 0$$

where the second inequality follows from the assumption that $P \in \mathscr{F}_4$. Moreover, because $(k, m_b, m_g, \mathbf{w}_b, \mathbf{w}_g) \in \Gamma(\mathbf{v}, g)$, we find

$$\begin{aligned} TP(\mathbf{v}, g) &\geq -k + p_g R(k) + \delta p_g [P(\mathbf{w}_g, g) - P(\mathbf{w}_b, b)] + \delta P(\mathbf{w}_b, b) \\ &\geq -k + p_b R(k) + \delta p_b [P(\mathbf{w}_g, g) - P(\mathbf{w}_b, b)] + \delta P(\mathbf{w}_b, b) \\ &= TP(\mathbf{v}, b) \end{aligned}$$

where the second line holds because $P(\mathbf{w}_g, g) - P(\mathbf{w}_b, b) \geq 0$ and $R(k) \geq 0$. Thus, $TP(\mathbf{v}, s) \in \mathscr{F}_4$ as claimed. □

Lemma 1.5. Let $\mathscr{F}_5 = \{P(\mathbf{v}, s) \in \mathscr{F} : P(\mathbf{v} + (\varepsilon, \varepsilon), s) \geq P(\mathbf{v}, s), \forall \varepsilon > 0\}$. Then $TP(\mathbf{v}, s) \in \mathscr{F}_5$. Moreover, $P(\mathbf{v}, s) \in \mathscr{F}_5$ implies there exists a policy with $m_b(\mathbf{v}, s) = 0$ that is optimal in [P1].

Proof. Let $(k, m_b, m_g, \mathbf{w}_b, \mathbf{w}_g)$ be the optimal policy at state $(\mathbf{v}, s)$. Since $(k, m_b - \varepsilon, m_g - \varepsilon, \mathbf{w}_b, \mathbf{w}_g) \in \Gamma(\mathbf{v} + (\varepsilon, \varepsilon), s)$ and m_b, m_g do not appear in the objective of [P1], we know $TP(\mathbf{v} + (\varepsilon, \varepsilon), s) \geq TP(\mathbf{v}, s)$. (Notice that the proof does not require $P \in \mathscr{F}_5$. It only requires that $P \in \mathscr{F}$, which ensures the existence of a maximiser.)

To prove the second claim, let us suppose $m_b < 0$, and consider the policy $(k', m'_b, m'_g, \mathbf{w}'_b, \mathbf{w}'_g) = (k, 0, m_g, \mathbf{w}_b - (m_b, m_b)/\delta, \mathbf{w}_g)$. It is easy to see that the new policy $(k', m'_b, m'_g, \mathbf{w}'_b, \mathbf{w}'_g) \in \Gamma(\mathbf{v}, s)$. Since $P \in \mathscr{F}_5$, we know $P(\mathbf{w}'_b, b) \geq P(\mathbf{w}_b, b)$. Hence the new policy $(k', m'_b, m'_g, \mathbf{w}'_b, \mathbf{w}'_g)$ at least weakly increases the objective of [P1]. Thus, there exists a policy with $m_b = 0$ that is optimal, completing the proof. □

Because the optimal contract lies in the interior of the feasible set (in an appropriate sense), the continuous differentiability of TP and P follows from standard results as, for instance, in **slp89**. To establish further properties of the value function, we will consider the optimality conditions for the problem [P1].

In what follows, $\hat{\eta}_b(\mathbf{v}, s)$ and $\hat{\eta}_g(\mathbf{v}, s)$ are the Lagrange multipliers for the promise keeping constraints [PK$_b$] and [PK$_g$] in [P1], $\hat{\alpha}(\mathbf{v}, s)$ is the Lagrange multiplier for the incentive constraint [IC], and $\hat{\mu}_b(\mathbf{v}, s)$ and $\hat{\mu}_g(\mathbf{v}, s)$ are the Lagrange multipliers for the limited liability constraints [LL] respectively. The first order conditions for

[P1] are

$$\text{[BFOC}k\text{]} \qquad R'(k(\mathbf{v},s)) = 1/[p_s - \hat{\eta}_g(\mathbf{v},s) + \hat{\mu}_g(\mathbf{v},s)]$$

$$\text{[BFOC}w_{bb}\text{]} \qquad (1-p_s)P_b(\mathbf{w}_b(\mathbf{v},s),b) = \hat{\eta}_b(\mathbf{v},s)(1-p_b) + \hat{\alpha}(\mathbf{v},s)(1-p_g)$$

$$\text{[BFOC}w_{bg}\text{]} \qquad (1-p_s)P_g(\mathbf{w}_b(\mathbf{v},s),b) = \hat{\eta}_b(\mathbf{v},s)p_b + \hat{\alpha}(\mathbf{v},s)p_g$$

$$\text{[BFOC}w_{gb}\text{]} \qquad p_s P_b(\mathbf{w}_g(\mathbf{v},s),g) = \hat{\eta}_g(\mathbf{v},s)(1-p_g) - \hat{\alpha}(\mathbf{v},s)(1-p_g)$$

$$\text{[BFOC}w_{gg}\text{]} \qquad p_s P_g(\mathbf{w}_g(\mathbf{v},s),g) = \hat{\eta}_g(\mathbf{v},s)p_g - \hat{\alpha}(\mathbf{v},s)p_g$$

The envelope conditions for [P1] are:

$$\text{[BEn}v_b\text{]} \qquad (TP)_b(\mathbf{v},s) = \hat{\eta}_b(\mathbf{v},s)$$

$$\text{[BEn}v_g\text{]} \qquad (TP)_g(\mathbf{v},s) = \hat{\eta}_g(\mathbf{v},s)$$

The following lemma establishes some properties of the Lagrange multipliers. Writing down the Lagrangean is straightforward and is hence omitted.

Lemma 1.6. For any $(\mathbf{v},s) \in V \times S$, the Lagrange multipliers in [P1] satisfy the following:

(a) The coefficient of m_b in the Lagrangian is $\hat{\eta}_b(\mathbf{v},s) + \hat{\alpha}(\mathbf{v},s) - \hat{\mu}_b(\mathbf{v},s) \geq 0$.

(b) The complementary slackness condition $m_b(\mathbf{v},s)[\hat{\eta}_b(\mathbf{v},s) + \hat{\alpha}(\mathbf{v},s) - \hat{\mu}_b(\mathbf{v},s)] = 0$.

(c) The coefficient of m_g in the Lagrangian is $\hat{\eta}_g(\mathbf{v},s) - \hat{\alpha}(\mathbf{v},s) - \hat{\mu}_g(\mathbf{v},s) = 0$.

Proof. To see parts (a) and (b), notice first that m_b, m_g only appear in the Lagrangian for [P1] in a linear way. The term multiplying m_b in the Lagrangian of [P1] is $\hat{\eta}_b + \hat{\alpha} - \hat{\mu}_b$. This term must be nonnegative. Otherwise, at $s = b$, maximizing the Lagrangian means that the optimal transfer is $m_b = -\infty$, since m_b is unbounded below. But this simply means the Lagrangian is unbounded above which is a contradiction since P is bounded above by $\bar{Q}$. Moreover, if $\hat{\eta}_b + \hat{\alpha} - \hat{\mu}_b > 0$ then maximizing the Lagrangian implies $m_b = 0$, because $m_b \leq 0$ by [LL]. Hence we always have $m_b(\hat{\eta}_b + \hat{\alpha} - \hat{\mu}_b) = 0$. These observations establish parts (a) and (b).

Now we show that part (c) holds. As m_g is unbounded below, using the observations in the paragraph above, we find that $\hat{\eta}_g - \hat{\alpha} - \hat{\mu}_g \geq 0$. Let $(k, m_b, m_g, \mathbf{w}_b, \mathbf{w}_g)$ be the optimal policy at any $(\mathbf{v},s)$. For any $\varepsilon > 0$, consider another policy:

$$(k', m'_b, m'_g, \mathbf{w}'_b, \mathbf{w}'_g) = \left(R^{-1}(R(k)+\varepsilon), m_b, m_g, \mathbf{w}_b, \mathbf{w}_g + \frac{\varepsilon}{\delta}(1,1)\right)$$

By definition,

$$\begin{aligned}
(TP)_g(\mathbf{v},s) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}[TP((v_b,v_g+\varepsilon),s) - TP((v_b,v_g),s)] \\
&\geq \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left\{p_s R(k') - k' - p_s R(k) + k + p_s\delta\left[P\left(\mathbf{w}'_g, g\right) - P(\mathbf{w}_g, g)\right]\right\} \\
&= p_s - \frac{1}{R'(k)} + p_s[P_b(\mathbf{w}_g,g) + P_g(\mathbf{w}_g,g)] \\
&= p_s - \frac{1}{R'(k)} + \hat{\eta}_g(\mathbf{v},s) - \hat{\alpha}(\mathbf{v},s) \qquad [1.5]
\end{aligned}$$

The inequality in the second line is because $(k', m'_b, m'_g, \mathbf{w}'_b, \mathbf{w}'_g) \in \Gamma((v_b, v_g + \varepsilon), s)$. The equality in the third line is L'Hôspital's rule for the second term, while the third term is the directional derivative of $P(\mathbf{w}_g, g)$ in the direction $(1,1)$. The last equality on the fourth line is obtained by adding [BFOCw_{gb}] and [BFOCw_{gg}]. Moreover,[BFOCk], [BEnv_g], and [1.5] together imply $\hat{\eta}_g(\mathbf{v},s) - \hat{\alpha}(\mathbf{v},s) - \hat{\mu}_g(\mathbf{v},s) \leq 0$. In conjunction with our earlier observation that $\hat{\eta}_g - \hat{\alpha} - \hat{\mu}_g \geq 0$, this proves part (c). □

Note that we can simplify [BFOCk] by using part (c) of Lemma 1.6 as,

$$R'(k(\mathbf{v},s)) = 1/[p_s - \hat{\alpha}(\mathbf{v},s)] \qquad [\text{BFOC}k^*]$$

Let us say that a function $f : (0,1) \to \mathbb{R}$ is *locally increasing* if for any $x \in (0,1)$, there exists $\varepsilon > 0$ such that f is increasing on $[x, x+\varepsilon]$. (Here, increasing is taken to mean non-decreasing.) The following is a useful lemma that shows that continuous locally increasing functions on an interval are increasing.

Lemma 1.7. Let $f : (0,1) \to \mathbb{R}$ be continuous and locally increasing. Then, f is increasing, ie, for all $x, y \in (0,1)$, $x \leq y$ implies $f(x) \leq f(y)$.

Proof. Let $x, y \in (0,1)$ such that $x < y$. We want to show that $f(x) \leq f(y)$. Because f is continuous, f achieves a maximum, M, on $[x,y]$. Moreover, because f is continuous, the set $Z := \{z \in [x,y] : f(z) = M\}$ is closed. Let z^* be the supremum of Z, which is in Z because Z is closed. If $z^* = y$, we are done, because $f(x) \leq M$. If, however, $z^* < y$, then by the hypothesis that f is locally increasing, there exists $\varepsilon > 0$ (which depends on z^*) such that f is increasing on $[z^*, z^* + \varepsilon] \subset [x,y]$. This implies $f(z^* + \varepsilon) = M$, because M is the maximum value of f on $[x,y]$, which contradicts the definition of z^*, namely that z^* is the supremum of the set Z and that $z^* < y$. □

The following lemma establishes that there exist optimal solutions to problem [P1] where the choice of $\mathbf{w}_g$ at $(\mathbf{v}, s)$ is independent of v_b.

Lemma 1.8. Take any $P \in \mathcal{F}$. In problem [P1], any $\mathbf{w}_g$ that is optimal at $(\mathbf{v}, s)$ with $\mathbf{v} \in [(0, v), (v, v)]$ is also optimal at $(\mathbf{v}', s)$ with any other $\mathbf{v}' \in [(0, v), (v, v)]$ for all $v \geq 0$. [2] In this sense, every optimal solution $\mathbf{w}_g(\mathbf{v}, s)$ is independent of v_b, s.

Proof. To proceed with the proof, consider the following auxiliary problem

$$\text{[P2]} \qquad \max_{x_g \geq x_b \geq 0} \delta P(\mathbf{x}, g) \quad \text{s.t.} \quad y \geq \delta\, \mathbb{E}^s[\mathbf{x}]$$

Because P is continuous, and because the set of feasible $\mathbf{x}$ in problem [P2] is compact, it follows that there exists an optimal solution, that we denote by $\mathbf{x}^*(y, s)$.

Fix any $(\mathbf{v}, s) \in V \times S$. Let $(k^*, m_i^*, \mathbf{w}_i^*)$ be a solution to [P1] at $(\mathbf{v}, s)$. Also, let $\mathbf{x}^*$ be a solution to [P2] at (v_g, s). We will show the following statements:

(a) $\mathbf{w}_g^*$ is a solution to [P2] at (v_g, s).
(b) There exists m_g such that the policy $(k^*, m_b^*, m_g, \mathbf{w}_b^*, \mathbf{x}^*)$ is a solution to [P1] at $(\mathbf{v}, s)$.

This will prove the lemma because every solution to [P2] is independent of v_b. So any $(k^*, m_i^*, \mathbf{w}_i^*)$ which is a solution to [P1] features that $\mathbf{w}_g^*$ is independent of v_b.

First, [PK_g] requires $v_g = R(k^*) - m_g^* + \delta\, \mathbb{E}^s[\mathbf{w}_g^*]$ while the limited liability constraint [LL] stipulates that $R(k^*) \geq m_g^*$. These two facts combine to give us $v_g \geq \delta\, \mathbb{E}^s[\mathbf{w}_g^*]$, which implies $\mathbf{w}_g^*$ is feasible in the auxiliary problem [P2] at (v_g, s). Optimality of $\mathbf{x}^*$ at (v_g, s) in [P2] implies that $P(\mathbf{x}^*, g) \geq P(\mathbf{w}_g^*, g)$.

Next, let $m_g := R(k^*) + \delta\, \mathbb{E}^s[\mathbf{x}^*] - v_g$. Notice that the constraint of [P2] implies that $m_g \leq R(k^*)$. So the policy $(k, m_b, m_g', \mathbf{w}_b, \mathbf{w}_g') \in \Gamma(\mathbf{v}, s)$, ie, it is feasible in problem [P1] at $(\mathbf{v}, s)$. But this feasible strategy cannot exhibit a higher value in problem [P1] than the optimal strategy $(k^*, m_i^*, \mathbf{w}_i^*)$, which implies that $P(\mathbf{w}_g^*, g) \geq P(\mathbf{x}^*, g)$.

This establishes that $P(\mathbf{w}_g^*, g) = P(\mathbf{x}^*, g)$. Because $\mathbf{w}_g^*$ is feasible in problem [P2] at (v_g, s), $\mathbf{w}_g^*$ has to be an optimal solution to [P2], thereby establishing (a). Moreover, $P(\mathbf{w}_g^*, g) = P(\mathbf{x}^*, g)$ also implies that $(k^*, m_b^*, m_g, \mathbf{w}_b^*, \mathbf{x}^*) \in \Gamma(\mathbf{v}, s)$ is a solution to [P1] at $(\mathbf{v}, s)$, which establishes (b). This concludes the proof. □

We now establish that the operator T preserves supermodularity in $\mathbf{v}$ for all s. Let $\mathcal{F}_6 = \{P(\mathbf{v}, s) \in \mathcal{F}_1 \cap \mathcal{F}_5 : P(\mathbf{v}, s) \text{ is supermodular}\}$.

Lemma 1.9. If $P(\mathbf{v}, s) \in \mathcal{F}_5$, then the policies $\mathbf{w}_b(\mathbf{v}, s)$ and $k(\mathbf{v}, s)$ satisfy

$$\text{[1.6]} \qquad w_{bb}(\mathbf{v}, s) \geq \frac{p_b R(k(\mathbf{v}, s)) + p_g v_b - p_b v_g}{\delta \Delta}$$

$$\text{[1.7]} \qquad w_{bg}(\mathbf{v}, s) \leq \frac{1 - p_b}{\delta \Delta} \left[v_g - \frac{1 - p_g}{1 - p_b} v_b - R(k(\mathbf{v}, s)) \right]$$

both of which are equalities if and only if [IC] holds as equality.

(2) Here, $[\mathbf{x}, y]$ denotes the closed interval connecting the vectors $\mathbf{x}$ and y, ie, $[\mathbf{x}, y] := \{t\mathbf{x} + (1 - t)y : t \in [0, 1]\}$.

Proof. Take any $P(\mathbf{v}, s) \in \mathscr{F}_6$. Let $(k, \mathbf{w}_i, m_i)$ be an optimal solution at $(\mathbf{v}, s)$. By Lemma 1.5, $m_b = 0$ is optimal. Then [PK$_b$] becomes

$$v_b = \delta\, \mathbb{E}^b[\mathbf{w}_b] \tag{1.8}$$

Using [1.8] and [IC], we obtain the two inequalities [1.6] and [1.7]. If [IC] holds as equality, then we have two equalities containing w_{bb}, w_{bg}. Solving w_{bb}, w_{bg} from [IC*] and [1.8] leads to [1.6] [1.7] both being equalities. If both [1.6] and [1.7] hold as equalities, then we can use them together with [1.8] to derive $v_g - v_b = R(k) + \delta\Delta(w_{bg} - w_{bb})$. So [IC] holds as equality. □

Lemma 1.10. If $P(\mathbf{v}, s) \in \mathscr{F}_6$, then $k(\mathbf{v}, s)$ decreases in v_b and increases in v_g.

Proof. Note that by [BFOCk*] it is equivalent to show that $\hat{\alpha}(\mathbf{v}, s)$ increases in v_b and decreases in v_g. Take some sufficient small $\varepsilon > 0$ and define $\mathbf{v}' = \mathbf{v} + (\varepsilon, 0)$. We show in the following that $\hat{\alpha}(\mathbf{v}', s) \geq \hat{\alpha}(\mathbf{v}, s)$. Then by Lemma 1.7, $\hat{\alpha}(\mathbf{v}, s)$ is increasing in v_b,

A trivial case is $\hat{\alpha}(\mathbf{v}, s) = 0$. Then the result simply follows because the Lagrange multiplier $\hat{\alpha}(\mathbf{v}', s) \geq 0$. So now suppose that $\hat{\alpha}(\mathbf{v}, s) > 0$ and $\hat{\alpha}(\mathbf{v}', s) < \hat{\alpha}(\mathbf{v}, s)$. These assumptions imply that [IC] holds as equality at $(\mathbf{v}, s)$ by complementary slackness and that $k(\mathbf{v}', s) > k(\mathbf{v}, s)$ by [BFOCk*].

Applying [1.6] in Lemma 1.9 at both $(\mathbf{v}, s)$ and $(\mathbf{v}', s)$, we obtain

$$\begin{aligned} w_{bb}(\mathbf{v}', s) &\geq \frac{p_b R(k(\mathbf{v}', s)) + p_g v_b' - p_b v_g}{\delta\Delta} \\ &> \frac{p_b R(k(\mathbf{v}, s)) + p_g v_b - p_b v_g}{\delta\Delta} = w_{bb}(\mathbf{v}, s) \end{aligned}$$

The strict inequality is because $k(\mathbf{v}', s) > k(\mathbf{v}, s)$ and $v_b' > v_b$, while the equality is because [IC] holds as an equality at $(\mathbf{v}, s)$. Similarly, applying [1.7] at $(\mathbf{v}, s)$ and $(\mathbf{v}', s)$ we obtain

$$\begin{aligned} w_{bg}(\mathbf{v}', s) &\leq \frac{1 - p_b}{\delta\Delta}\left[v_g - \frac{1 - p_g}{1 - p_b} v_b' - R(k(\mathbf{v}', s))\right] \\ &< \frac{1 - p_b}{\delta\Delta}\left[v_g - \frac{1 - p_g}{1 - p_b} v_b - R(k(\mathbf{v}, s))\right] = w_{bg}(\mathbf{v}, s) \end{aligned}$$

Once again, the strict inequality is because $k(\mathbf{v}', s) > k(\mathbf{v}, s)$ and $v_b' > v_b$, while the equality is because [IC] holds as equality at $(\mathbf{v}, s)$.

Because $P_g(\mathbf{v}, s)$ is increasing v_b (recall that $P \in \mathscr{F}_6$) and because P_g is decreasing in v_g (recall that by $P \in \mathscr{F}_6 \subset \mathscr{F}_1$ is concave in $\mathbf{v}$), we have $P_g(\mathbf{w}_b(\mathbf{v}', s), b) \geq P_g(\mathbf{w}_b(\mathbf{v}, s), b)$. This implies

$$\begin{aligned} \eta_b(\mathbf{v}', s) &= (1 - p_s) P_g(\mathbf{w}_b(\mathbf{v}', s), b) - p_g \hat{\alpha}(\mathbf{v}', s) \\ &> (1 - p_s) P_g(\mathbf{w}_b(\mathbf{v}, s), b) - p_g \hat{\alpha}(\mathbf{v}, s) \\ &= \eta_b(\mathbf{v}, s) \end{aligned}$$

where the first and last equalities obtain from [BFOCw_{bg}], while the strict inequality follows because, as noted above, $P \in \mathcal{F}_6$ and because $\hat{\alpha}(\mathbf{v}', s) < \hat{\alpha}(\mathbf{v}, s)$ (by assumption). The envelope condition [BEnv$_b$] now implies $(TP)_b(\mathbf{v}', s) > (TP)_b(\mathbf{v}, s)$. However, from Lemma 1.1, $TP(\mathbf{v}, s) \in \mathcal{F}_1$ and therefore concave in $\mathbf{v}$, which requires that $(TP)_b(\mathbf{v}', s) \leq (TP)_b(\mathbf{v}, s)$, a contradiction. Hence $\hat{\alpha}(\mathbf{v}', s) \geq \hat{\alpha}(\mathbf{v}, s)$.

Since we can use the same procedure in the above argument to show $\alpha(\mathbf{v}, s)$ decreases in v_g, we do not repeat. □

Lemma 1.11. If $P(\mathbf{v}, s) \in \mathcal{F}_6$, then $TP(\mathbf{v}, s) \in \mathcal{F}_6$.

Proof. Note that the supermodularity of $TP(\mathbf{v}, s)$ is equivalent to $TP_g(\mathbf{v}, s)$ being increasing in v_b. Let $\varepsilon > 0$ be sufficient small and define $\mathbf{v}' = \mathbf{v} + (\varepsilon, 0)$.

By Lemma [1.8], the left hand side of [BFOCw_{gg}] is constant on $[\mathbf{v}, \mathbf{v}']$, which implies $\eta_g - \hat{\alpha}$ is constant on this interval. Having established that $\hat{\alpha}(\mathbf{v}', s) \geq \hat{\alpha}(\mathbf{v}, s)$ in Lemma 1.10, this implies $\hat{\eta}_g(\mathbf{v}', s) \geq \hat{\eta}_g(\mathbf{v}, s)$, which further implies $(TP)_g(\mathbf{v}', s) \geq (TP)_g(\mathbf{v}, s)$ by [BEnv$_g$]. So $(TP)_g(\mathbf{v}, s)$ is locally increasing in v_b. By Lemma 1.7, $(TP)_g(\mathbf{v}, s)$ is therefore increasing in v_b, and so $TP(\mathbf{v}, s) \in \mathcal{F}_6$. □

Theorem 1. *The unique fixed point of T, which we call Q, lies in $\bigcap_{i=1}^{6} \mathcal{F}_i$. Therefore, Q satisfies the following:*

(a) *$Q(\mathbf{v}, s)$ is concave in $\mathbf{v}$,*
(b) *$Q(\mathbf{v}, s) = \bar{Q}(s)$ for any $\mathbf{v} \in E'_s$,*
(c) *$Q(\mathbf{v}, s)$ is decreasing in v_b at $((v, v), s)$ for any $v > 0$,*
(d) *$Q(\mathbf{v}, g) \geq Q(\mathbf{v}, b)$,*
(e) *$Q(\mathbf{v} + (\varepsilon, \varepsilon), s) \geq Q(\mathbf{v}, s)$,*
(f) *the optimal investment $k(\mathbf{v}, s)$ decreases in v_b and increases in v_g, and*
(g) *$Q_g(\mathbf{v}, s)$ is supermodular.*

Proof. It is easy to see that T is monotone (whereby $P_1 \leq P_2$ implies $TP_1 \leq TP_2$) and satisfies discounting (wherein $T(P + a) = TP + \delta a$), which implies T is a contraction mapping on $\mathcal{F}$ and hence has a unique fixed point in $\mathcal{F}$. We have established (in Lemmas 1.1 through 1.11) that if $P \in \bigcap_{i=1}^{6} \mathcal{F}_i$, then $TP \in \bigcap_{i=1}^{6} \mathcal{F}_i$. This implies the unique fixed point of T also lies in $\bigcap_{i=1}^{6} \mathcal{F}_i$. □

2. Strict Concavity of the Surplus Function

In this section we show that $Q(\mathbf{v}, s)$ is strictly concave on the set H. It suffices to restrict attention to the set H because the optimal contract always initiates and stays in the set H until the firm becomes unconstrained as shown in Proposition **??** of the text.

Let h^t be a period t public history, as described in Section **??**. Let $k^t(h^{t-1};\mathbf{v},s)$ be the optimal investment induced by the optimal contract in period t when the optimal contract begins at state $(\mathbf{v},s)$ and has history h^{t-1}. Similarly, let $\mathbf{w}_i^t(h^{t-1};\mathbf{v},s)$ be the optimal contingent utilities generated from policy functions starting at state $(\mathbf{v},s)$ after history h^{t-1}. The following is a variant of Lemma 11.3.1 in **mailath-samuelson-book06**.

Lemma 2.1. For $(\tilde{\mathbf{v}},s),(\hat{\mathbf{v}},s)\in V\times S$ with $\tilde{\mathbf{v}}\neq\hat{\mathbf{v}}$, and $\theta\in(0,1)$, if $k^t(h^{t-1};\hat{\mathbf{v}},s)\neq k^t(h^{t-1};\tilde{\mathbf{v}},s)$ for some history h^{t-1}, then $Q(\theta\hat{\mathbf{v}}+(1-\theta)\tilde{\mathbf{v}},s)>\theta Q(\hat{\mathbf{v}},s)+(1-\theta)Q(\tilde{\mathbf{v}},s)$.

Proof. Let us change the control variable k in the original programming [**??**] by letting $r=R(k)$, so that $C(r):=R^{-1}(r)=k$. Then $C(\cdot)$ is strictly convex and all the constraints are linear in r. Define the average policies as

$$
\begin{aligned}
\bar{k}^t(h^{t-1}) &= \theta k^t(h^{t-1};\hat{\mathbf{v}},s)+(1-\theta)k^t(h^{t-1};\tilde{\mathbf{v}},s)\\
\bar{\mathbf{w}}_i^t(h^{t-1}) &= \theta \mathbf{w}_i^t(h^{t-1};\hat{\mathbf{v}},s)+(1-\theta)\mathbf{w}_i^t(h^{t-1};\tilde{\mathbf{v}},s)
\end{aligned}
$$

To ease notation, let $\hat{r}^t(h^{t-1})=r^t(h^{t-1};\hat{\mathbf{v}},s)$, $\tilde{r}^t(h^{t-1})=r^t(h^{t-1};\tilde{\mathbf{v}},s)$, $\hat{\mathbf{w}}_i^t(h^{t-1})=\mathbf{w}_i^t(h^{t-1};\hat{\mathbf{v}},s)$, and $\tilde{\mathbf{w}}_i^t(h^{t-1})=\mathbf{w}_i^t(h^{t-1};\tilde{\mathbf{v}},s)$. Now, consider the payoffs obtained from following the optimal contract for T periods, and then obtaining the value contained in the value function when starting from states $(\hat{\mathbf{v}},s)$ and $(\tilde{\mathbf{v}},s)$ respectively.

$$
\begin{aligned}
Q(\hat{\mathbf{v}},s) =& \sum_{t=0}^{T}\delta^t\mathbb{E}_0[-C(\hat{r}^t(h^{t-1}))+p_{s^{t-1}}\hat{r}^t(h^{t-1})]\\
&+\delta^{T+1}\mathbb{E}_0[p_{s^{T-1}}Q(\hat{\mathbf{w}}_g^T(h^{T-1}),g)+(1-p_{s^{T-1}})Q(\hat{\mathbf{w}}_b^T(h^{T-1}),b)]
\end{aligned}
$$

$$
\begin{aligned}
Q(\tilde{\mathbf{v}},s) =& \sum_{t=0}^{T}\delta^t\mathbb{E}_0[-C(\tilde{r}^t(h^{t-1}))+p_{s^{t-1}}\tilde{r}^t(h^{t-1})]\\
&+\delta^{T+1}\mathbb{E}_0[p_{s^{T-1}}Q(\tilde{\mathbf{w}}_g^T(h^{T-1}),g)+(1-p_{s^{T-1}})Q(\tilde{\mathbf{w}}_b^T(h^{T-1}),b)]
\end{aligned}
$$

Averaging for large enough T, we obtain

$$
\begin{aligned}
&\theta Q(\hat{\mathbf{v}}, s) + (1-\theta) Q(\tilde{\mathbf{v}}, s) \\
&\quad = \sum_{t=0}^{T} \delta^t \mathbb{E}_0[-\theta C(\hat{r}^t(h^{t-1})) - (1-\theta) C(\tilde{r}^t(h^{t-1})) + p_{s^{t-1}} \bar{r}^t(h^{t-1})] \\
&\qquad + \theta \delta^{T+1} \mathbb{E}_0[p_{s^{T-1}} Q(\hat{\mathbf{w}}_g^T(h^{T-1}), g) + (1 - p_{s^{T-1}}) Q(\hat{\mathbf{w}}_b^T(h^{T-1}), b)] \\
&\qquad + (1-\theta) \delta^{T+1} \mathbb{E}_0[p_{s^{T-1}} Q(\tilde{\mathbf{w}}_g^T(h^{T-1}), g) + (1 - p_{s^{T-1}}) Q(\tilde{\mathbf{w}}_b^T(h^{T-1}), b)] \\
&\quad < \sum_{t=0}^{T} \delta^t \mathbb{E}_0[-C(\bar{r}^t(h^{t-1})) + p_{s^{t-1}} \bar{r}^t(h^{t-1})] \\
&\qquad + \delta^{T+1} \mathbb{E}_0[p_{s^{T-1}} Q(\bar{\mathbf{w}}_g^T(h^{T-1}), g) + (1 - p_{s^{T-1}}) Q(\bar{\mathbf{w}}_b^T(h^{T-1}), b)] \\
&\quad \le Q(\theta \hat{\mathbf{v}} + (1-\theta) \tilde{\mathbf{v}}, s)
\end{aligned}
$$

The strict equality follows from the facts that $\hat{r}^t(h^{t-1}) \neq \tilde{r}^t(h^{t-1})$ for some history h^{t-1} and that the residual term is close to zero after a large enough T. The weak equality follows from the fact that the average plan $(\bar{r}^t, \bar{\mathbf{w}}_i^t)_{t=0}^T$ satisfies the constraints of the Bellman equation at every step of the iteration starting from state $(\theta \hat{\mathbf{v}} + (1-\theta) \tilde{\mathbf{v}}, s)$. □

Corollary 2.2. If $(\tilde{\mathbf{v}}, s), (\hat{\mathbf{v}}, s) \in V \times S$ are such that $Q(\tilde{\mathbf{v}}, s) \neq Q(\hat{\mathbf{v}}, s)$, then Q is strictly concave on $[(\hat{\mathbf{v}}, s), (\tilde{\mathbf{v}}, s)]$, the line segment joining $(\hat{\mathbf{v}}, s)$ and $(\tilde{\mathbf{v}}, s)$.

Proof. Because $Q(\tilde{\mathbf{v}}, s) \neq Q(\hat{\mathbf{v}}, s)$, we know there must exist a history h^{t-1} such that $k^t(h^{t-1}; \hat{\mathbf{v}}, s) \neq k^t(h^{t-1}; \tilde{\mathbf{v}}, s)$. The result now follows immediately from Lemma 2.1, because otherwise, the firm's surplus will be the same starting at both $(\tilde{\mathbf{v}}, s)$ and $(\hat{\mathbf{v}}, s)$. □

Lemma 2.3. For any $(\mathbf{v}, s) \in H$, $Q(\mathbf{v}, s)$ is strictly concave in both v_b and v_g.

Proof. Take any $(\tilde{\mathbf{v}}, s), (\hat{\mathbf{v}}, s) \in H$ with $\tilde{v}_g < \hat{v}_g$ and $\tilde{v}_b = \hat{v}_b$. Then $Q(\hat{\mathbf{v}}, s) > Q(\tilde{\mathbf{v}}, s)$, because $Q_g(\cdot, s) > 0$ and Q is concave. The result follows immediately from Corollary 2.2. The same argument shows $Q(\mathbf{v}, s)$ is strictly concave in v_b for $(\mathbf{v}, s) \in H$. □

Lemma 2.4. For any $(\mathbf{v}, s) \in H$ and $v_g \ge \bar{y}_g$, investment $k(\mathbf{v}, s)$ decreases in v_b, and strictly increases in v_g.

Proof. Take any $(\mathbf{v}, s) \in H$ and $v_g \ge \bar{y}_g$. The left hand side of [**??**] is zero, because $\mathbf{w}_g(\mathbf{v}, s) \in E_g$ by Lemma **??** of the Appendix. Moreover, $\eta_g(\mathbf{v}, s) = Q_g(\mathbf{v}, s)$ increases in v_b, and strictly decreases in v_g. The former is by the supermodularity of Q and the latter is by Lemma 2.3. Then we can see from [**??**] that $\alpha(\mathbf{v}, s)$ increases in v_b, and strictly decreases in v_g. From [**??**], $k(\mathbf{v}, s)$ decreases in v_b, and strictly increases in v_g. □

Proposition 2.5. For $(\tilde{\mathbf{v}}, s), (\hat{\mathbf{v}}, s) \in H$ with $\tilde{\mathbf{v}} \neq \hat{\mathbf{v}}$ and $\theta \in (0, 1)$, $Q(\theta\hat{\mathbf{v}} + (1 - \theta)\tilde{\mathbf{v}}, s) > \theta Q(\hat{\mathbf{v}}, s) + (1 - \theta)Q(\tilde{\mathbf{v}}, s)$.

Proof. Take any $(\tilde{\mathbf{v}}, s), (\hat{\mathbf{v}}, s) \in H$ with $\tilde{\mathbf{v}} \neq \hat{\mathbf{v}}$. By Lemma 2.1, it suffices to show that $k^t(h^{t-1}; \hat{\mathbf{v}}, s) \neq k^t(h^{t-1}; (\tilde{\mathbf{v}}, s))$ for some history h^{t-1}. Suppose not. The firm surplus and investment must be the same after any history starting at the two initial states $(\tilde{\mathbf{v}}, s)$ and $(\hat{\mathbf{v}}, s)$. This means we must have either $\tilde{v}_b > \hat{v}_b$ and $\tilde{v}_g < \hat{v}_g$, or $\tilde{v}_b < \hat{v}_b$ and $\tilde{v}_g > \hat{v}_g$. Otherwise, $Q(\hat{\mathbf{v}}, s) \neq Q(\tilde{\mathbf{v}}, s)$, because $Q_b(\cdot, s), Q_g(\cdot, s) > 0$ on H. Without loss of generality, we assume $\tilde{v}_b > \hat{v}_b, \tilde{v}_g < \hat{v}_g$. At history $h^0 = \{s, g\}$, we know $Q(\hat{\mathbf{w}}_g^0(h^0), g) = Q(\tilde{\mathbf{w}}_g^0(h^0), g)$, which implies $\Psi(\hat{v}_g, g) = \Psi(\tilde{v}_g, g)$ in Lemma **??** of the Appendix. If $\tilde{v}_g < \bar{y}_g$, then Lemma **??** of the Appendix implies that we must have $\hat{v}_g = \tilde{v}_g$, since $\Psi(y, g)$ is increasing overall and strictly increasing in $y \in [0, \delta[p_g\bar{v}_{gg} + (1 - p_g)\bar{v}_{gb}])$, a contradiction. If $\tilde{v}_g \geq \bar{y}_g$, then $k(\hat{\mathbf{v}}, s) > k((\hat{v}_b, \tilde{v}_g), s) \geq k(\tilde{\mathbf{v}}, s)$ by Lemma 2.4. This forms a contradiction. □

3. Other Properties of the Surplus Function

Lemma 3.1. For any $v \geq 0$ and $s \in S$, firm surplus function satisfies $Q(\mathbf{0}, s) = 0$, $Q_g((v, v), s) = \infty$, and $Q_b((0, v), s) = \infty$.

Proof. Because the only feasible policy at state $(\mathbf{0}, s)$ is

$$(k, m_b, m_g, \mathbf{w}_b, \mathbf{w}_g) = (0, 0, 0, \mathbf{0}, \mathbf{0})$$

we must have $Q(\mathbf{0}, s) = 0$.

We first show that $Q_g(\mathbf{0}, s) = \infty$, and $\mathsf{D}_{(1,1)}\, Q(\mathbf{0}, s) = \infty$. Then we use these two facts to show $Q_b((0, v), s) = \infty$. Note that

$$\begin{aligned} Q_g(\mathbf{0}, s) &= \lim_{\varepsilon \to 0} \frac{Q((0, \varepsilon), g)}{\varepsilon} \\ &\geq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[-R^{-1}(\varepsilon) + p_s\varepsilon + \delta p_s Q\left((0, \frac{\varepsilon}{\delta p_g}), g \right) \right] \\ &= p_s + \frac{p_s}{p_g} Q_g(\mathbf{0}, g) \qquad [3.1] \end{aligned}$$

The inequality is because

$$(k, m_b, m_g, \mathbf{w}_b, \mathbf{w}_g) = \left(R^{-1}(\varepsilon), 0, \varepsilon, \mathbf{0}, \left(0, \frac{\varepsilon}{\delta p_g}\right) \right) \in \Gamma((0, \varepsilon), s)$$

Because $p_g > 0$, we must have $Q_g(\mathbf{0}, g) = \infty$ for [3.1] to hold when $s = g$. Then $Q_g(\mathbf{0}, b) = \infty$ is implied by [3.1] when $s = b$.

Next,

$$\mathsf{D}_{(\mathbf{1},\mathbf{1})}\, Q(\mathbf{0}, s) = \lim_{\varepsilon\to 0} \frac{\mathsf{D}_{(\mathbf{1},\mathbf{1})}\, Q((\varepsilon,\varepsilon), s)}{\varepsilon}$$

$$\geq \lim_{\varepsilon\to 0} \frac{\delta}{\varepsilon}\left[p_s Q\left((0, \frac{\varepsilon}{\delta p_g}), g\right) + (1-p_s) Q\left((\frac{\varepsilon}{\delta}, \frac{\varepsilon}{\delta}), b\right)\right]$$

[3.2] $$= \frac{p_s}{p_g} Q_g(\mathbf{0}, g) + (1-p_s)\,\mathsf{D}_{(\mathbf{1},\mathbf{1})}\, Q(\mathbf{0}, b)$$

The inequality is because

$$(k, m_b, m_g, \mathbf{w}_b, \mathbf{w}_g) = \left(0, 0, 0, \left(\frac{\varepsilon}{\delta}, \frac{\varepsilon}{\delta}\right), \left(0, \frac{\varepsilon}{\delta p_g}\right)\right) \in \Gamma((\varepsilon,\varepsilon), b)$$

Because $Q_g(\mathbf{0}, g) = \infty$, evaluating $\mathsf{D}_{(\mathbf{1},\mathbf{1})}\, Q(\mathbf{0}, b)$ using [3.2] implies $\mathsf{D}_{(\mathbf{1},\mathbf{1})}\, Q(\mathbf{0}, b) = \infty$. Then evaluating $\mathsf{D}_{(\mathbf{1},\mathbf{1})}\, Q(\mathbf{0}, g)$ using [3.2] implies $\mathsf{D}_{(\mathbf{1},\mathbf{1})}\, Q(\mathbf{0}, g) = \infty$.

Next, let $(k, 0, m_g, \mathbf{0}, \mathbf{w}_g)$ be the optimal policy at $((0, v), s)$ for any $v > 0$. The optimal $m_b = w_{bi} = 0$ is implied by [PK_b] at $v_b = 0$. Then we have

$$\left(k, 0, m_g, \left(\frac{\varepsilon}{\delta}, \frac{\varepsilon}{\delta}\right), \mathbf{w}_g\right) \in \Gamma((\varepsilon, v), s)$$

which implies

$$Q_b((0, v), s) = \lim_{\varepsilon\to 0} \frac{Q((\varepsilon, v), s) - Q((0, v), s)}{\varepsilon}$$

$$\geq \lim_{\varepsilon\to 0} \frac{(1-p_s)\delta}{\varepsilon}\left[Q\left(\left(\frac{\varepsilon}{\delta}, \frac{\varepsilon}{\delta}\right), b\right) - Q(\mathbf{0}, b)\right]$$

$$= (1-p_s)\,\mathsf{D}_{(\mathbf{1},\mathbf{1})}\, Q(\mathbf{0}, b) = \infty$$

Next, let's define $Q_b((v, v), s)$ as the left derivative since only $v_g \geq v_b$ is feasible. For arbitrary small $\varepsilon > 0$ and $v' = v - \frac{1-p_b}{1-p_g} R(\varepsilon)$, we get

$$Q((v', v), b) - Q((v, v), b)$$

[3.3] $$\geq -\varepsilon + p_b R(\varepsilon) + (1-p_b)\delta\left[Q\left(\left(w_{bg}, w\right), b\right) - Q((w, w), b)\right] \geq 0$$

where $w = \frac{v+m_b}{\delta}$, and $w_{bg} = w - \frac{R(\varepsilon)}{\delta(1-p_g)}$. Divide both sides of [3.3] by $\frac{1-p_b}{1-p_g} R(\varepsilon)$, and take the limit as ε converges to zero to obtain:

[3.4] $$-Q_b((v, v), b) \geq \frac{p_b(1-p_g)}{1-p_b} - Q_b((w, w), b)$$

Because $Q_b((w, w), s) \leq 0$ by Theorem 1 (c), we know $-Q_b((v, v), b) \geq \frac{p_b(1-p_g)}{1-p_b}$. The same argument shows that $-Q_b((w, w), b) \geq \frac{p_b(1-p_g)}{1-p_b}$. So $-Q_b((v, v), b) \geq \frac{2p_b(1-p_g)}{1-p_b}$ by [3.4]. Repeating this procedure, we get the result $-Q_b((v, v), b) \geq \frac{n p_b(1-p_g)}{1-p_b}$ for any $n \in \mathbb{N}$. Hence, we must have $Q_b((v, v), b) = -\infty$.

Now let $(k, m_b, m_g, (w, w), \mathbf{w}_g)$ be the optimal policy at $((v, v), s)$ for some $v > 0$, where $w = \frac{v+m_b}{\delta}$. Also let $w'_{bb} = w - \varepsilon$, $m'_b = m_b - (1 - p_b)\delta\varepsilon$, and $m'_g = m_g - \varepsilon$. Then we have

$$(k, m'_b, m'_g, (w - \varepsilon, w), \mathbf{w}_g) \in \Gamma((v, v + \varepsilon), s)$$

So optimality at $((v, v + \varepsilon), s)$ implies

$$\begin{aligned} Q_g((v, v), s) &\geq \lim_{\varepsilon \to 0} \frac{\delta(1 - p_s)}{\varepsilon} \left[Q\left((w - \varepsilon, w), b\right) - Q((w, w), b)\right] \\ &= -\delta(1 - p_s) Q_b((w, w), b) = \infty \end{aligned}$$

□

Using the first order conditions, we can characterize the evolution of the directional derivative martingale $\mathsf{D}_{(1,1)}\, Q$ on any path induced by the optimal contract, as in the following Lemma.

Lemma 3.2. For the shock pairs 'good-good', 'good-bad', 'bad-good', and 'bad-bad' and for an initial state $(\mathbf{v}, s)$, let the states induced by the optimal contract at $(\mathbf{v}, s)$ be $\mathbf{w}_i = \mathbf{w}_i(\mathbf{v}, s)$, and let the states induced by the optimal contract at $(\mathbf{w}_g, g)$ and $(\mathbf{w}_b, b)$ be $\mathbf{w}_i^g = \mathbf{w}_i(\mathbf{w}_g, g)$ and $\mathbf{w}_i^b = \mathbf{w}_i(\mathbf{w}_b, b)$ respectively. Then, the martingale $\mathsf{D}_{(1,1)}\, Q$ evolves according to the following relations

[3.5] $$\mathsf{D}_{(1,1)}\, Q(\mathbf{w}_g^g, g) = \mathsf{D}_{(1,1)}\, Q(\mathbf{w}_g, g) - \frac{1}{p_g}\alpha(\mathbf{w}_g, g)$$

[3.6] $$\mathsf{D}_{(1,1)}\, Q(\mathbf{w}_b^g, b) = \mathsf{D}_{(1,1)}\, Q(\mathbf{w}_g, g) + \frac{1}{1 - p_g}\alpha(\mathbf{w}_g, g)$$

[3.7] $$\mathsf{D}_{(1,1)}\, Q(\mathbf{w}_g^b, g) = \mathsf{D}_{(1,1)}\, Q(\mathbf{w}_b, b) - \frac{(1 - p_s)\alpha(\mathbf{w}_b, b) - \Delta\alpha(\mathbf{v}, s)}{p_b(1 - p_s)}$$

[3.8] $$\mathsf{D}_{(1,1)}\, Q(\mathbf{w}_b^b, b) = \mathsf{D}_{(1,1)}\, Q(\mathbf{w}_b, b) + \frac{(1 - p_s)\alpha(\mathbf{w}_b, b) - \Delta\alpha(\mathbf{v}, s)}{(1 - p_b)(1 - p_s)}$$

Proof. First, from [**??**] and [**??**], we get:

[3.9] $$\begin{aligned} p_s[\eta_g(\mathbf{w}_g, g) - \alpha(\mathbf{w}_g, g)] &= p_s Q_g(\mathbf{w}_g, g) - p_s\alpha(\mathbf{w}_g, g)] \\ &= p_g[\eta_g(\mathbf{v}, s) - \alpha(\mathbf{v}, s)] - p_s\alpha(\mathbf{w}_g, g) \end{aligned}$$

Add [**??**] and [**??**] at state $(\mathbf{v}, s)$ to get:

[3.10] $$p_s\, \mathsf{D}_{(1,1)}\, Q(\mathbf{w}_g, g) = \eta_g(\mathbf{v}, s) - \alpha(\mathbf{v}, s)$$

Add [**??**] and [**??**] at state $(\mathbf{w}_g, g)$ to get:

[3.11] $$p_g\, \mathsf{D}_{(1,1)}\, Q(\mathbf{w}_g^g, g) = \eta_g(\mathbf{w}_g, g) - \alpha(\mathbf{w}_g, g)$$

Then combine [3.9], [3.10], [3.11] and rearrange to get [3.5].

Next, from [??] and [??], we get

[3.12] $$\begin{aligned}&(1-p_s)[\eta_b(\mathbf{w}_b, b) + \alpha(\mathbf{w}_b, b)]\\ &= (1-p_s)Q_b(\mathbf{w}_b, b) + (1-p_s)\alpha(\mathbf{w}_b, b)\\ &= (1-p_b)[\eta_b(\mathbf{v}, s) + \alpha(\mathbf{v}, s)] - \Delta\alpha(\mathbf{v}, s) + (1-p_s)\alpha(\mathbf{w}_b, b)\end{aligned}$$

Add [??] and [??] at state $(\mathbf{v}, s)$ to get

[3.13] $$(1-p_s)\,\mathsf{D}_{(\mathbf{1},\mathbf{1})}\,Q(\mathbf{w}_b, b) = \eta_b(\mathbf{v}, s) + \alpha(\mathbf{v}, s)$$

Add [??] and [??] at state $(\mathbf{w}_b, b)$ to get

[3.14] $$(1-p_b)\,\mathsf{D}_{(\mathbf{1},\mathbf{1})}\,Q(\mathbf{w}_b^b, b) = \eta_b(\mathbf{w}_b, b) + \alpha(\mathbf{w}_b, b)$$

Then combine [3.12],[3.13], [3.14] and rearrange to get [3.8].

Finally, equations [3.5] and [3.6] together imply

[3.15] $$(1-p_g)\,\mathsf{D}_{(\mathbf{1},\mathbf{1})}\,Q(\mathbf{w}_b^g, b) + p_g\,\mathsf{D}_{(\mathbf{1},\mathbf{1})}\,Q(\mathbf{w}_g^g, g) = \mathsf{D}_{(\mathbf{1},\mathbf{1})}\,Q(\mathbf{w}_g, g)$$

Equations [3.7] and [3.8] together imply

[3.16] $$(1-p_b)\,\mathsf{D}_{(\mathbf{1},\mathbf{1})}\,Q(\mathbf{w}_b^b, b) + p_b\,\mathsf{D}_{(\mathbf{1},\mathbf{1})}\,Q(\mathbf{w}_g^b, g) = \mathsf{D}_{(\mathbf{1},\mathbf{1})}\,Q(\mathbf{w}_b, b)$$

Then we can combine [3.5] [3.15] and rearrange to obtain [3.6]. Similarly, we can combine [3.8] [3.16] and rearrange to obtain [3.7]. □

Lemma 3.3. For any $\mathbf{v} \in V$ with $v_b \geq \bar{v}_{sb}$, $Q_b(\mathbf{v}, s) \leq 0$.

Proof. Note that by concavity of Q it suffices to show that $Q_b((\bar{v}_{sb}, v_g), s) \leq 0$. When $v_g \geq \bar{v}_{sg}$, we know $(\bar{v}_{sb}, v_g) \in E_s$. Lemma **??** in the text simply implies $Q_b((\bar{v}_{sb}, v_g), s) = 0$. When $v_g < \bar{v}_{sg}$, by the supermodularity of Q, we know that $Q_b((\bar{v}_{sb}, v_g), s) \leq Q_b((\bar{v}_{sb}, \bar{v}_{sg}), s) = 0$. □

Lemma 3.4. Let $G := \{(\mathbf{v}, s) \in V \times S : \mathbf{v} < \bar{\mathbf{v}}_s\}$, and let $\mathrm{cl}(H)$, $\mathrm{cl}(G)$ be the closure of set H and set G respectively. Then,

(a) $H \subset G$;

(b) For any $s \in S$, $(\bar{\mathbf{v}}_s, s) \in \mathrm{cl}(H)$;

(c) For any $(\mathbf{v}, s) \in \mathrm{cl}(H)$, $Q_g(\mathbf{v}, s) + Q_b(\mathbf{v}, s) = 0$ implies $\mathbf{v} = \bar{\mathbf{v}}_s$.

Proof. (a) Take any $(\mathbf{v}, s) \in H$. By definition $Q_b(\mathbf{v}, s), Q_g(\mathbf{v}, s) > 0$. So Lemma 3.3 implies $v_b < \bar{v}_{sb}$. Suppose $v_g \geq \bar{v}_{sg}$. Then

$$0 \leq Q_g(\mathbf{v}, s) \leq Q_g((\bar{v}_{sb}, v_g), s) = 0$$

The first inequalities is because Q_g is positive. The second inequality is because of the supermodularity of Q. The last equality is because $(\bar{v}_{sb}, v_g) \in E_s$ which implies $Q_g((\bar{v}_{sb}, v_g), s) = 0$. Hence, $Q_g(\mathbf{v}, s) = 0$, a contradiction.

(b) Part (a) implies $\mathrm{cl}(H) \subseteq \mathrm{cl}(G)$. Suppose $(\bar{\mathbf{v}}_s, s) \notin \mathrm{cl}(H)$ for some $s \in \{b, g\}$. Then there must exist some $\varepsilon > 0$ such that $N_\varepsilon \cap \mathrm{cl}(H) = \varnothing$ where $N_\varepsilon := \{(\mathbf{v}, s) : \bar{\mathbf{v}}_s - (\varepsilon, \varepsilon) \leq \mathbf{v} < \bar{\mathbf{v}}_s\}$. However, there must exist $\eta > 0$ such that $(x, f_s(x)) \in N_\varepsilon$ for any $x \in [\bar{v}_{sb} - \eta, \bar{v}_{sb})$. The function $f_s(\cdot)$ is defined in [**??**] of the Appendix. The existence of η is because $f_s(\cdot)$ is increasing and satisfies $f_s(\bar{v}_{sb}) = \bar{v}_{sg}$ by Lemma **??** of the Appendix. Moreover, by the definition of $\bar{v}_{sb}$ (see [**??**] of the Appendix), $Q_b((x, f_s(x)), s) > 0$ for all such x. Then the continuity of $Q_b(\cdot, s)$ imply that there exists some region slightly below the curve $\{\mathbf{v} : v_g = f_s(v_b), v_b \in [\bar{v}_{sb} - \eta, \bar{v}_{sb})\}$ and locates in N_ε that satisfies $Q_g(\mathbf{v}, s) > 0, Q_b(\mathbf{v}, s) > 0$. This forms a contradiction. Hence, $(\bar{\mathbf{v}}_s, s) \in \mathrm{cl}(H)$.

(c) Take any $(\hat{\mathbf{v}}, s) \in \mathrm{cl}(H)$ such that $\mathsf{D}_{(1,1)} Q(\hat{\mathbf{v}}, s) = 0$. Then $\hat{\mathbf{v}} \leq \bar{\mathbf{v}}_s$ and $Q_b(\hat{\mathbf{v}}, s) = Q_g(\hat{\mathbf{v}}, s) = 0$. By the definition of E_s, we know $\hat{\mathbf{v}} \in E_s$, and hence, $\hat{\mathbf{v}} \geq \bar{\mathbf{v}}_s$. So we must have $\hat{\mathbf{v}} = \bar{\mathbf{v}}_s$. □

Lemma 3.5. For any $\mathbf{v} \in V$ with $v_g \geq R(\bar{k}_s) + \frac{p_g v_b}{p_b}$, $k(\mathbf{v}, s) = \bar{k}_s$.

Proof. Take any $\mathbf{v} \in V$ with $v_g \geq R(\bar{k}_s) + \frac{p_g v_b}{p_b}$. Then $v_g - v_b \geq R(\bar{k}_s) + \frac{\Delta v_b}{p_b}$. By [PK$_b$] at $(\mathbf{v}, s)$,

$$\begin{aligned} &\delta[p_b(w_{bg}(\mathbf{v}, s) - w_{bb}(\mathbf{v}, s)) + w_{bb}(\mathbf{v}, s)] \leq v_b \\ \Longrightarrow\ & w_{bg}(\mathbf{v}, s) - w_{bb}(\mathbf{v}, s) \leq \frac{v_b}{\delta p_b} \\ \Longrightarrow\ & R(\bar{k}_s) + \delta\Delta(w_{bg} - w_{bb}) \leq R(\bar{k}_s) + \frac{\Delta v_b}{p_b} \leq v_g - v_b \end{aligned}$$

The second line is because $w_{bb}(\mathbf{v}, s) \geq 0$. The last line implies it's always feasible to invest efficiently, ie $(\bar{k}_s, m_i(\mathbf{v}, s), \mathbf{w}_i(\mathbf{v}, s)) \in \Gamma(\mathbf{v}, s)$. Hence, $k(\mathbf{v}, s) = \bar{k}_s$. □

Lemma 3.6. For any $v_b \geq 0$, there exists $h_s(v_b) > v_b$ such that $v_g \geq h_s(v_b)$ implies $k(\mathbf{v}, s) = \bar{k}_s$, and $v_b \leq v_g < h_s(v_b)$ implies $k(\mathbf{v}, s) < \bar{k}_s$. Moreover, $h_s(v_b)$ is increasing and satisfies $h_s(0) = R(\bar{k}_s), h_s(\bar{v}_{sb}) \leq \bar{v}_{sg}$;

Proof. Take any $v_b > 0$. If v_g is sufficiently close to v_b, then investment $k(\mathbf{v}, s)$ will be sufficiently close to zero by [IC*]. If v_g is sufficiently large, by Lemma 3.5, we know $k(\mathbf{v}, s) = \bar{k}_s$. Hence, for a certain value of v_b we can define

$$h_s(v_b) = \min\{\mathbf{v} \in V : v_g \geq v_b, k(\mathbf{v}, s) = \bar{k}_s\} \qquad [3.17]$$

Since $k(\mathbf{v}, s)$ is increasing in v_g by Theorem 1 (f), we know that $k(\mathbf{v}, s) = \bar{k}_s$ when $v_g \geq h_s(v_b)$.

Consider $(\mathbf{v}, s) = ((0, R(\bar{k}_s)), s)$. Since the only feasible $\mathbf{w}_b$ at $(\mathbf{v}, s)$ is $\mathbf{0}$ and $v_g = R(\bar{k}_s)$, we know $k(\mathbf{v}, s) = \bar{k}_s$. Since $k(\bar{\mathbf{v}}_s, s) = \bar{k}_s$, we know $h_s(\bar{v}_b) \leq \bar{v}_{sg}$. Now we show $h_s(v_b)$ is increasing. Take any v_b, v_b' with $v_b' > v_b \geq 0$. Suppose

$h_s(v_b') < h_s(v_b)$. Then there exists $\varepsilon > 0$ such that $h_s(v_b') < h_s(v_b) - \varepsilon$. By the definition of $h_s(\cdot)$,

$$k[(v_b', h_s(v_b')), s] = \bar{k}_s > k[(v_b, h_s(v_b) - \varepsilon), s] \geq k[(v_b', h_s(v_b) - \varepsilon), s]$$

The last line is because $k(\mathbf{v}, s)$ is decreasing in v_b by Theorem 1 (f). Then we must have $h_s(v_b') > h_s(v_b) - \varepsilon$. Otherwise, $k[(v_b', h_s(v_b')), s] \leq k[(v_b', h_s(v_b) - \varepsilon), s]$ because $k(\mathbf{v}, s)$ is increasing in v_g. This is a contradiction. Hence, $h_s(v_b') \geq h_s(v_b)$ □

Lemma 3.7. For any $(\mathbf{v}, s) \in H$ such that $v_g \geq h_s(v_b)$, $Q_b(\mathbf{v}, s) = Q_b[(v_b, h_s(v_b), s]$ and $Q_g(\mathbf{v}, s) = Q_g[(0, v_g), s]$.

Proof. Denote $\mathbf{v}' = (v_b, h_s(v_b))$. Since $v_g \geq h_s(v_b) \geq h_s(0)$ by the monotonicity of $h_s(\cdot)$, we know $\alpha(\mathbf{v}, s) = \alpha(\mathbf{v}', s) = \alpha[(0, v_g), s] = 0$. Lemma **??** of the Appendix implies $\mathbf{w}_b(\mathbf{v}', s) = \mathbf{w}_b(\mathbf{v}, s)$, which implies the left hand side of [3.13] is the same at state $(\mathbf{v}, s)$ or $(\mathbf{v}', s)$. Hence [3.13] implies $Q_b(\mathbf{v}, s) = Q_b(\mathbf{v}', s)$.

Similarly, Lemma **??** of the Appendix implies $\mathbf{w}_g(\mathbf{v}, s) = \mathbf{w}_g[(0, v_g), s]$. So the left hand side of [3.10] is the same at state $(\mathbf{v}, s)$ or $\big((0, v_g), s\big)$. Hence [3.10] implies $Q_g(\mathbf{v}, s) = Q_g[(0, v_g), s]$. □

Lemma 3.8. Fix $s \in S$. Then, $Q_b[(v_b, h_s(v_b)), s]$ strictly decreases in v_b on the set $[0, \delta\, \mathbb{E}^b[\bar{\mathbf{v}}_b])$.

Proof. Take any $v_b < \hat{v}_b \leq \delta\, \mathbb{E}^b[\bar{\mathbf{v}}_b]$. And let $\mathbf{v} = (v_b, h_s(v_b))$ and $\hat{\mathbf{v}} = (\hat{v}_b, h_s(\hat{v}_b))$. By the definition of $h_s(\cdot)$, $\alpha(\mathbf{v}, s) = \alpha(\hat{\mathbf{v}}, s) = 0$. Then from Lemma **??** of the Appendix, we know that $\mathbf{w}_b(\mathbf{v}, s), \mathbf{w}_b(\hat{\mathbf{v}}, s)$ are solutions to problem [**??**] of the Appendix at (v_b, b), $(\hat{v}_b, b)$ respectively. In addition, by [3.13] we have

$$(1 - p_s)\Psi_y(v_b, b) = (1 - p_s)\,\mathsf{D}_{(\mathbf{1},\mathbf{1})}\, Q[\mathbf{w}_b(\mathbf{v}, s), b] = Q_b(\mathbf{v}, s)$$
$$(1 - p_s)\Psi_y(\hat{v}_b, b) = (1 - p_s)\,\mathsf{D}_{(\mathbf{1},\mathbf{1})}\, Q[\mathbf{w}_b(\hat{\mathbf{v}}, s), b] = Q_b(\hat{\mathbf{v}}, s)$$

By Lemma **??** of the Appendix, $\Psi(\cdot, b)$ is strictly concave on the set $[0, \delta\, \mathbb{E}^b[\bar{\mathbf{v}}_b])$. Then the above equations imply that $Q_b(\hat{\mathbf{v}}, s) < Q_b(\mathbf{v}, s)$. □

4. Some Pathwise Properties of Markov Chains

In this section, we collect some miscellaneous facts about paths of Markov chains. Let E be the (countable) state space for a Markov process with transition probabilities $P(x, B)$ denoting the probability of transitioning from x to B. Let $\mathbf{P}$ denote the induced probability measure on the path space E^∞.

Lemma 4.1. Let (X_n) be an E-valued Markov process with transitions given by the kernel P, and suppose $x \in E$ is *recurrent*. Then,

$$\mathbf{P}(X_n = x \text{ for infinitely many } n \mid X_0 = x) = 1$$

This is a standard proposition in the theory of Markov chains and has many proofs (including one via the Ergodic Theorem for Markov chains). An elementary proof can be found on p 577 of **shiryaev-prob-text**.

Proposition 4.2. The event $\{(s_{t-1}, s_t) = (i, j) \text{ for infinitely many } t\}$ occurs almost surely for all $i, j \in S$. Therefore, the event $\{s_t = i \text{ for infinitely many } t\}$ occurs almost surely for all $i \in S$.

(Here, 'almost surely' is with respect to the probability measure induced on S^∞ by the Markov kernel.) Intuitively, the proposition says that bad or good shocks occur infinitely often with probability one. Moreover, consecutive 'bad-bad', 'bad-good', 'good-bad', and 'good-good' shocks also occur infinitely often with probability one.

Proof of Proposition 4.2. Recall that $S = \{b, g\}$ with transition probabilities $P(i, j)$ represented by the transition matrix

$$\begin{array}{c} \\ b \\ g \end{array} \begin{array}{c} \begin{array}{cc} b & g \end{array} \\ \begin{pmatrix} 1 - p_b & p_b \\ 1 - p_g & p_g \end{pmatrix} \end{array}$$

As the statement of the lemma notes, it suffices to prove that pairs of shocks of the form (i, j) occur infinitely along almost every sample path.

Towards the proof of this claim, it is useful to consider the *bivariate* Markov chain with states $E = \{bb, bg, gb, gg\}$ and transition probabilities Q given by

$$\begin{array}{c|cccc} & bb & bg & gb & gg \\ \hline bb & 1 - p_b & p_b & 0 & 0 \\ bg & 0 & 0 & 1 - p_g & p_g \\ gb & 1 - p_b & p_b & 0 & 0 \\ gg & 0 & 0 & 1 - p_g & p_g \end{array}$$

The transition matrix reflects the fact that one-step transitions have a simple form, namely

$$Q\big((i, j), (k, \ell)\big) = \mathbf{1}\{j = k\} P(k, \ell)$$

where $\mathbf{1}\{j = k\} = 1$ if $j = k$ and 0 otherwise. Then, the two-step transition probabilities are given by

$$Q^{(2)}\big((i, j), (k, \ell)\big) = P(j, k) P(k, \ell) > 0$$

Therefore, all states communicate with each other, which implies that the Markov chain is indecomposable. But because the state space E is finite, by Theorem 1 (and the subsequent discussion) on p 580 of **shiryaev-prob-text**, at least one of the states must be recurrent. The indecomposability of the process then implies that all states are recurrent. An application of Lemma 4.1 completes the proof. □