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A geometric approach to continuous expected utility ☆

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Abstract

Inspired by the recent literature on preferences over menus, we present alternate axiomatisations of the Expected Utility Theorem. These new axioms also lead to a new proof.

Keywords: Expected utility; Independence; Translation invariance

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1. Introduction

The Expected Utility Theorem is the cornerstone of axiomatic choice under uncertainty. However, the traditional Independence axiom is not always intuitive. This is especially true in a setting where an agent has preferences over menus. We consider here some axioms that are arguably more intuitive and yet are equivalent to Independence for continuous preferences. These axioms also lead us to an intuitive proof based on the notion that translation invariant indifference classes lead to contour sets that can be separated by hyperplanes.

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2. Axioms and theorem

Let $(\mathcal{X})$ be a topological vector space and $C \subset \mathcal{X}$ be metrisable. Typical elements of C will be denoted by x, y, z and p, q, r etc. The zero vector in $(\mathcal{X})$ will be denoted by θ and let d be a metric generating the topology. We are interested in preference relations on C, ie binary relations $\succcurlyeq \subset C \times C$ where $\succcurlyeq$ is complete and transitive. We shall restrict attention to preferences that are continuous which is equivalent to saying that the sets $\{q: q \succ p\}$ and $\{q: p \succ p\}$ are open for each $p \in C$. The indifference class of a lottery p will be denoted by $[p]$. The ε-neighbourhood of a point q (in C) will be denoted by $N_\varepsilon(q)$. Therefore, in the present context, continuity amounts to saying that for any $p, q \in C$ such that $p \succ q$, there exists $\varepsilon > 0$ small enough so that for any $p' \in N_\varepsilon(p)$ and $q' \in N_\varepsilon(q)$, $p' \succ q'$. Our formulation is general enough to encompass two important examples.

Example 2.1. Let Z be a compact metric space and $\mathcal{P}Z$ be the space of Borel probability measures on Z, $\mathcal{M}(Z)$ the space of signed Borel measures on Z and $\mathcal{C}(Z)$ the space of continuous functions on Z. Then, $\langle \mathcal{C}(Z), \mathcal{M}(Z) \rangle$ is a dual pair and endowing $\mathcal{M}(Z)$ with the weak* topology renders it metrisable and $\mathcal{P}(Z)$ compact. This is the canonical domain of expected utility theory.

Example 2.2. Let Z be a set of $n+1$ prizes, Δ^n, the n-dimensional simplex, be the space of probability measures on Z and $\mathcal{A}$ the space of compact convex subsets of Δ^n. When endowed with the Hausdorff metric, $\mathcal{A}$ becomes a compact metric space. Furthermore, it can be shown that $\mathcal{A}$ can be embedded isometrically as a compact convex subset of $\mathcal{C}(S^{n-1})$, the Banach space of continuous functions (when endowed with the sup norm) on the $(n-1)$-dimensional sphere. Preferences over sets of lotteries are explored in Dekel et al. (2001), where a proof of the embedding can be found. Such preferences are used to model flexibility and commitment, and more generally in the construction of a subjective state space. Ahn (2005) uses preferences over lotteries to study ambiguity.

The following are additional axioms we will impose.

Axiom 1. Independence

$p \succ q$ and $\lambda \in (0, 1]$ imply $\lambda p + (1-\lambda) r \succ \lambda q + (1-\lambda) r$.

Axiom 2. Betweenness

(i) $p \succ q$ and $\lambda \in (0, 1)$ imply $p \succ \lambda p + (1-\lambda) q \succ q$ and (ii) $p \sim q$ and $\lambda \in (0, 1)$ imply $p \sim \lambda p + (1-\lambda) q$.

Axiom 3. Convex contour sets

The sets $\{q: q \succ p\}$, $\{q: p \succ q\}$ and $[p]$ are convex.

Axiom 4. Translation invariance

$p \succcurlyeq q$ implies $p + c \succcurlyeq q + c$ for all $c \in (\mathcal{X})$ such that $p + c, q + c \in C$.

Axiom 5. Sure-thing principle for lotteries

For all $p, q, r, s \in C$ and $\lambda \in (0, 1)$, $\lambda p + (1-\lambda) r \succcurlyeq \lambda q + (1-\lambda) r$ if and only if $\lambda p + (1-\lambda) s \succcurlyeq \lambda q + (1-\lambda) s$.

The axiom *Betweenness* was first introduced by Dekel (1986) in a model of preferences under uncertainty without *Independence*. *Translation invariance* is used by Ahn (2005) and Chatterjee and Krishna (2007) (both in different contexts) and the proof below is based on the latter's approach. In the context of preferences over lotteries, and especially as it pertains to ambiguity, *Translation invariance* seems quite natural. Consider the following example.

Example 2.3. Consider an agent who has to choose between two medical procedures. Suppose the first procedure is relatively new and has a probability of success anywhere between [0.2, 0.4]. Suppose there is another, more established, procedure that has 0.3 probability of success. Here, a set of lotteries represents ambiguity in the sense that the agent considers all probabilities in the set as likely to be the true probability. If the agent prefers the first procedure (which has probabilities [0.2, 0.4]) to the second procedure, it is reasonable to expect him to prefer a procedure with a set of probabilities given by [0.3, 0.5] to the procedure which succeeds with probability 0.4. This is the essence of *Translation invariance*. *Translation invariance* has also been used by Noor (2006) as an axiom in a model of temptation.

Finally, the counterpart to Savage's Sure-thing principle is the *Sure-thing principle for lotteries*.[1] This is clearly more appealing than *Independence*.

Definition 2.4. A preference relation $\succcurlyeq$ over C has an *Expected Utility (EU) representation* if there exists a continuous function $V: C \to R$ that satisfies $V(\lambda x+(1-\lambda)y)=\lambda V(x)+(1-\lambda)V(y)$ for all $x, y \in C$, $\lambda \in (0, 1)$.

Main theorem. *For any preference relation $\succcurlyeq$ over C, the following are equivalent:*

(a) $\succcurlyeq$ satisfies Continuity and Independence.
(b) $\succcurlyeq$ satisfies Continuity, Betweenness and Translation invariance.
(c) $\succcurlyeq$ satisfies Continuity and Translation invariance.
(d) $\succcurlyeq$ has an EU representation, unique up to positive affine transformation.
(e) $\succcurlyeq$ satisfies Continuity and Sure-thing principle.

Consider again the following special case.

Example 2.5. Let $C = \mathcal{P}(Z)$, where Z is compact metrisable. Then, there exists $v: Z \to R$, continuous, so that $V(p)=\int v(z)\mathrm{d}p(z)$. To see this, we take, as before, V continuous linear that represents $\succcurlyeq$. Define $v(z) := V(\delta_z)$ (where δ_z is the Dirac measure at z). It is easy to see that v is continuous. Define the linear functional $W(p) := \int v(z)\mathrm{d}p(z)$. Clearly, for any $p \in \mathcal{P}_S(Z)$, the space of all measures with finite support, we see that $V(p)=W(p)$. Since $V=W$ on a dense subset of $\mathcal{P}(Z)$ and since V is uniformly continuous, it follows that $V(p)=W(p)$ for all $p \in \mathcal{P}(Z)$.

3. Proofs

We provide a brief sketch of the proofs. If $p \sim q$ for all $p, q \in C$, we can let V be a constant function. We shall henceforth assume that there exist $p, q \in C$ so that $p \succ q$. Since it is immediate that *Independence* → *Betweenness* → *Convex contour sets*, Lemma 3.1 demonstrates that (a) → (b) → (c). Lemma 3.5 then

[1] We thank the anonymous referee for suggesting this axiom.

shows that (c) is equivalent to (a). It is easy to see that *Independence* implies the *Sure-thing principle*, ie (a) implies (e). Proposition 3.6 shows that (e) implies (c) and hence (a). To complete the proof of the theorem, we use the fact *Convex contour sets* to show that preferences can be represented by a linear functional on any finite subset. We fix two elements and their values and then calibrate the value of any other point in the domain. This gives a straightforward proof of the EU theorem.

Lemma 3.1. Translation invariance

Let $\succcurlyeq$ satisfy Independence and Continuity. Then $\succcurlyeq$ is translation invariant.

Proof. Let $p \succcurlyeq q$ and c such that $p+c$, $q+c \in C$. Simple geometry shows that for all $\lambda \in (0, 1)$,

$$\lambda q + (1-\lambda)(p+c) = \lambda\{\lambda q + (1-\lambda)(q+c)\} + (1-\lambda)\{\lambda p + (1-\lambda)(p+c)\}.$$

Then, since $\succcurlyeq$ is reflexive

$$\lambda q + (1-\lambda)(p+c) \sim \lambda\{\lambda q + (1-\lambda)(q+c)\} + (1-\lambda)\{\lambda p + (1-\lambda)(p+c)\}.$$

From *Independence* we get

$$\lambda p + (1-\lambda)(p+c) \succcurlyeq \lambda q + (1-\lambda)(p+c).$$

Combining the relations above

$$\lambda p + (1-\lambda)(p+c) \succcurlyeq \lambda\{\lambda q + (1-\lambda)(q+c)\} + (1-\lambda)\{\lambda p + (1-\lambda)(p+c)\}.$$

From *Independence* we see that

$$\lambda p + (1-\lambda)(p+c) \succcurlyeq \lambda q + (1-\lambda)(q+c).$$

From *Continuity* it now follows that $p+c \succcurlyeq q+c$. □

We shall see below that $(c) \rightarrow (d)$. Let us first show that for non-trivial preferences, *Translation invariance* implies indifference curves cannot be thick and then show that *Translation invariance* and *Continuity* implies *Betweenness*.

Lemma 3.2. Thin indifference curves

Let $\succcurlyeq$ satisfy Translation invariance and let $p \succ q \succ r$. Then for all $\varepsilon > 0$, there exist $p', r' \in C$ such that $p' \succ q \succ r'$ and $p', r' \in N_\varepsilon(q)$.

Proof. We shall show that p' exists with the desired properties. A symmetric argument shows that r' exists. Let us fix $\varepsilon > 0$. Now $p \succ q$. Define $p_1 := (p+q)/2$ and $c := p_1 - q$. To see that $p_1 \succ q$, let us suppose the contrary, i.e. $q \succcurlyeq p_1$. Then, $q \succcurlyeq p_1 = q + c \succcurlyeq p_1 + c = p$ where the middle relation follows from *Translation invariance* giving us the desired contradiction. Now let $p_n := (p_{n-1} + q)/2$ for all n. A simple induction shows that for all n, $p_n \succ q$. For n large enough, $d(p_n, q) < \varepsilon$. Let p' be such a p_n. □

Lemma 3.3. *Let $\succcurlyeq$ satisfy Translation invariance and Continuity. Then $\succcurlyeq$ satisfies Betweenness.*

Proof. Let $p \succ q$ and let $\lambda \in (0, 1)$. For each $n \in N$, define $p_0^n := p$ and $p_{i+1}^n := p_i^n + (q-p)/2^n$ for all $i = 0, \ldots, 2^n - 1$. By *Translation invariance*, $p \succ q$ implies $p \succ (p+q)/2 \succ q$ and repeated application gives $p = p_0^n \succ p_1^n \succ \cdots \succ p_{2n}^n = q$. Also, there exists $j \in \{0, \ldots, 2^n - 1\}$ such that $d(p_j^n, \lambda p + (1-\lambda)q) \leqslant d(p_k^n, \lambda p +$

$(1-\lambda)q$) for all $k\in\{0,\ldots,2^n-1\}$. Define the sequence $(r_n)_0^\infty$ where $r_n:=p_j^n$ so that $r_n\to\lambda p+(1-\lambda)q$. Since $p\succ r_n\succ q$ for each n and p, q are bounded away from $\lambda p+(1-\lambda)q$, from *Continuity* it follows that $p\succ\lambda p+(1-\lambda)q\succ q$. A similar argument shows that $p\sim q$ implies $p\sim\lambda p+(1-\lambda)q$ for all $\lambda\in(0,1)$. □

A useful corollary of *Betweenness* is the following.

Corollary 3.4. *For any p, q, $r\in C$, the following hold:*

(a) $p\succ q$ and $0\leqslant\mu<\lambda\leqslant1$ imply $\lambda p+(1-\lambda)q\succ\mu p+(1-\mu)q$.
(b) $p\succ q\succ r$ implies there exists a unique $\lambda^*\in(0,1)$ such that $\lambda^*p+(1-\lambda^*)q\sim r$.

Proof. We shall only use *Betweenness* in this proof.

(a) By *Betweenness*, $p\succ q$ and $0<\lambda<1$ imply $p\succ\lambda p+(1-\lambda)q\succ q$. Let $r:=\lambda p+(1-\lambda)q$ and notice that $\alpha:=(\mu/\lambda)<1$. Then, *Betweenness* once again implies that $r\succ\alpha r+(1-\alpha)q=(\mu/\lambda)r+(1-(\mu/\lambda))q=(\mu/\lambda)(\lambda p+(1-\lambda)q)+(1-(\mu/\lambda))q=\mu p+(1-\mu)q$.
(b) This is a straightforward consequence of *Continuity* and the result above. See, for instance, Lemma 5.6, pp. 67, Kreps (1988). □

We now show that *Continuity* and *Translation invariance* imply *Independence*.

Lemma 3.5. *Let $\succcurlyeq$ satisfy Continuity and Translation invariance. Then $\succcurlyeq$ satisfies Independence.*

Proof. Let p, q, $r\in C$, $p\succ q$ and $c:=p-q$. By Lemma 3.3, we know that $\succcurlyeq$ satisfies *Betweenness* so that $q+\lambda c=\lambda p+(1-\lambda)q\succ q$ for all $\lambda\in(0,1)$. Now let $c':=\lambda q+(1-\lambda)r-q=q(\lambda-1)+(1-\lambda)r$. Thus, *Translation invariance* implies $q+\lambda c+c'\succ q+c'$. But $q+\lambda c+c'=\lambda p+(1-\lambda)q+q(\lambda-1)+(1-\lambda)r=\lambda p+(1-\lambda)r$ and $q+c'=\lambda q+(1-\lambda)r$ so that $\lambda p+(1-\lambda)r\lambda q+(1\text{-}\lambda)r$ which proves that $\succcurlyeq$ satisfies *Independence*. □

To prove the equivalence of all the axioms, we now show that for continuous preferences, the *Sure-thing principle* implies *Translation invariance*.

Proposition 3.6. *Let $\succcurlyeq$ satisfy Continuity and Sure-thing principle. Then $\succcurlyeq$ satisfies Translation invariance.*

Proof. Suppose not, so that there exist p, $q\in C$ and $c\in(\mathcal{X})$ such that $p+c$, $q+c\in C$, $p\succcurlyeq q$ and $q+c\succ p+c$. We claim that for any $\varepsilon>0$, we can take c such that $p\in N_\varepsilon(p+c)$ and $q\in N_\varepsilon(q+c)$. Suppose this is true, then we have a contradiction since *Continuity* implies that for all $q'\in N_\varepsilon(q+c)$ and $p'\in N_\varepsilon(p+c)$, $q'\succ p'$ (ie it is not the case that $p'\succcurlyeq q'$) which contradicts $p\succcurlyeq q$. To prove the claim, it will suffice to show that if there exist p, q, c as above, then there exists p', $q'\in C$ and $c/2\in(\mathcal{X})$ such that $p'\succcurlyeq q'$ and $q'+c/2\succ p'+c/2$.

Notice that the *Sure-thing principle* implies $p+c/2\succcurlyeq(p+q)/2+c/2$ if and only if $(p+q)/2+c/2\succcurlyeq q+c/2$. Thus exactly one of the following must hold: (i) $p+c/2\succcurlyeq(p+q)/2+c/2\succcurlyeq q+c/2$, or (ii) $q+c/2\succ(p+q)/2+c/2\succ p+c/2$. Thus, either $p\succcurlyeq q$ and $q+c/2\succ p+c/2$ or $p\succcurlyeq q$ and $p+c/2\succcurlyeq q+c/2$ but not both. In the first case, let $p':=p$ and $q':=q$ Then, $p\succcurlyeq q$, but $q+c/2\succ p+c/2$. In the second case, let $p':=p+c/2$ and $q':=q+c/2$, so that $p'\succcurlyeq q'$ and $q'+c/2p'+c/2$. Thus, by continuing this process, can take c to be arbitrarily small, i.e. $p\in N_\varepsilon(p+c)$ and $q\in N_\varepsilon(q+c)$ for any $\varepsilon>0$, which completes the proof. □

To prove the Main theorem, notice that our domain C lives in a vector space. To prove the theorem, we assume $\theta\in C$. (To see that this is without loss of generality, see below.) We fix another element $p^\circ\in C$ and consider finite dimensional convex subsets that contain θ and p°. We show that for finite dimensional subsets, *Translation invariance* implies the EU theorem for the restricted domain. This is a simple application of the separating hyperplane theorem. Using this, we find that we can assign a value to any $p\in C$ in such a way that linearity is respected.

We can assume, without loss of generality, that $\theta \in C$. (If not, define $C'{:} = C - q$ for some $q \in C$ and look at the induced preferences.) Now fix $p^\circ \in C$ such that $p^\circ \nsim \theta$. (Assume, without loss of generality, that $p^\circ \succ \theta$.) We shall now show that for a class of finite dimensional subsets of C, with representative element D, that contain θ and p°, preferences restricted to D (i.e. $\succcurlyeq|_D$) have an EU representation. For any $D \subset C$, define $\mathcal{F}_D := (span\ D) \cap\ C$.

Lemma 3.7. *Let $D \subset C$ contain p°, θ and be a finite set. Then, there exists a linear functional f_D : span $D \rightarrow \mathbb{R}$ such that $f_D|_{\mathcal{F}_D}$ represents $\succcurlyeq|_D$ and has $f_D(\theta) = 0$, $f_D(p^\circ) = 1$.*

Proof. Since D is finite and $\theta, p^\circ \in D$, the convex hull of D, conv D, is finite dimensional, as is $\mathcal{F}_D$. Let $q_D \in ri\ \mathcal{F}_D$, the relative interior of $\mathcal{F}|_D$. By *Betweenness*, $\{q' \in C\colon q' \succ q_D\}$, $\{q' \in C\colon q_D \succ q'\}$ and $[q_D]$ are disjoint convex sets and by Lemma 3.2, $[q_D]$ is thin. Then, there exists a linear functional f_D : *span* $D \rightarrow \mathbb{R}$ such that $f_D(\theta) = 0$, $f_D(p^\circ) = 1$, f_D separates the upper and lower contour sets and $f(q) > f(q_D)$ for any $q \in \{q' \in C{:}q' \succ q_D\}$. Moreover, we claim that f_D represents $\succcurlyeq|_D$. □

To see this, suppose not, ie suppose there exist $q_1, q_2 \in \mathcal{F}_D$ such that $q_1 \succcurlyeq q_2$ but $f_D(q_1) > f_D(q_2)$. By *Translation invariance*, for any $\varepsilon > 0$, we can take $d(q_1, q_2) < \varepsilon$. Fix $\varepsilon > 0$ such that $N_\varepsilon(q_D) \cap \mathcal{F}_D \subset\ ri\ \mathcal{F}_D$. Then, we can take $c{:} = q_D - q_2$ so that $q_2 + c = q_D$ and $q_1 + c \in\ ri\ \mathcal{F}_D$. By *Translation invariance*, we have $q_1 + c \succcurlyeq q_2 + c$, ie $f_D(q_1 + c) \geqslant f_D(q_2 + c)$ which holds if and only if $f_D(q_1) + f_D(c) \geqslant f_D(q_2) + f_D(c)$ (since f_D is linear on all of span D), which is a contradiction. □

We can now proceed to the proof of the Main theorem.

Proof of Main theorem. For any $p \in C$, there exists a finite $D \subset C$ such that $p, p^\circ, \theta \in D$. Define $V(p){:} = f_D(p)$. Such a definition is valid if $f_D(p)$ is independent of the choice of D. Suppose this is the case, then it follows that $p \mapsto V(p)$ is linear and continuous, which proves the theorem.

We now show that $V(p)$ is independent of the choice of D. First suppose that $p \in \text{span}\{\theta, p^\circ\}$. Then, $p = \lambda p^\circ$ for some $\lambda \in R$. Then, for any D, $f_D(p) = f_D(\lambda p^\circ) = \lambda f_D(p^\circ) = \lambda$. Similarly, if $p \sim p^\circ$, then $f_D(p) = 1$ and $p \sim \theta$ implies $f_D(p) = 0$ for any choice of D. Now, if $p^\circ \succ p \succ \theta$, by Corollary 3.4 there is a unique $\lambda^* \in (0, 1)$ such that $p \sim \lambda^* p^\circ + (1 - \lambda^*)\theta$ so that for any choice of D, $f_D(p) = \lambda^* f_D(p^\circ) = \lambda^*$. Similarly, if $p \succ p^\circ$, by Corollary 3.4, there exists a unique $\lambda^* \in (0, 1)$ such that $p^\circ \sim \lambda^* p + (1 - \lambda^*)\theta$. Then, $f_D(p^\circ) = f_D(\lambda^* p)$, ie $f_D(p) = 1/\lambda^*$ for any choice of D. The case where $\theta \succ p$ is taken care of in a similar fashion. □

References

Ahn, D., 2005, Ambiguity without a State Space, mimeo, University of California, Berkeley.

Chatterjee, K., Krishna, R.V., 2007. Menu Choice, Environmental Cues and Temptation: A "Dual-Self" Approach to Self-control. Working Paper.

Dekel, E., 1986. An axiomatic characterization of preferences under uncertainty: weakening the independence axiom. Journal of Economic Theory 40, 304–318.

Dekel, E., Lipman, B.L., Rustichini, A., 2001. Representing preferences with a unique subjective state space. Econometrica 69 (4), 891–934.

Kreps, D.M., 1988. Notes on the Theory of Choice. Westview Press, Co., U.S.A.

Noor, J., 2006. Menu-dependent Self-control. working paper. Boston University.