# **Dynamic Procurement and Relational Capital***

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**Abstract**

We study a setting in which a principal wishes to procure a single unit of a good in each period, $t = 0, 1, 2, \ldots$. She may procure from the market at known cost $d$ or from an *inside* agent, who has a privately known cost of production. The agent's cost in each period is the realization of an iid draw from a continuous distribution $F(\cdot)$. The agent is risk-neutral but liquidity constrained and so must be advanced his cost of production by the principal in every period.

We formulate the resulting dynamic mechanism design problem as a recursive program in which the agent's promised utility at any history, $v$, is the relevant state variable that is naturally interpreted as his *relational capital*. In contrast with the static problem, we show: first that even for the uniform distribution, the monotonicity constraint on output will always hold with equality for a positive measure of types until his relational capital is sufficiently high; and second that the optimal allocation is not generally characterized by a cost cutoff, ie, may involve probabilistic procurement.

We prove that the agent eventually builds up enough relational capital $v^*$ to become a vested partner in the enterprise. Prior to attaining $v^*$ the agent is incentivised exclusively through adjustments to his relational capital, while after achieving $v^*$ he receives rents paid in cash. Upon becoming a vested partner, the agent stays a vested partner for ever. Furthermore, in every period, he is then given a right of first refusal to produce the good for a cash payment of $d$, so there are no more output distortions, and consequently, the monotonicity constraint no longer binds.

*Key Words:* Procurement, bunching, ironing, relational capital.

*JEL Classifications:* C61, D82, D86, L26

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# 1. Introduction

We study a setting in which a principal wishes to procure a single unit of a good in each period, $t = 0, 1, 2, \ldots$. She has two potential sources of supply: she may procure from the *market* or from an *inside* agent. The cost of procuring from the market is $d \in (0, 1]$ in each period. The agent may produce a unit of the good in period $t$ at a privately known cost $c_t \in [0, 1]$ which is drawn from a distribution $F$ and is iid over time.

The agent is risk-neutral but liquidity constrained. Specifically, he has no wealth of his own and must be advanced the cost of production at the beginning of any period in which the principal procures the good from him. This gives rise to an incentive problem in which the agent must be dissuaded from over-stating the cost of production and pocketing the difference between his reported and actual cost.

We formulate the resulting dynamic mechanism design problem as a recursive program in which the agent's promised utility $v_t$ at any time $t$ is the relevant state variable that is naturally interpreted as his *relational capital* with the principal.[1] Recall that in the static version of the problem, the optimal contract calls for the principal to offer the agent a take-it-or-leave-it offer (ie, bang-bang procurement) of

(1) There is a growing body of evidence on the importance of relational capital in supply chain management (eg, cost and quality control). See, for example, Cousins et al. (2006) and Krause, Handfield and Tyler (2007).

a payment of $c_*$ so that if the agent accepts, he will produce the good in exchange of a payment of $c_*$, and if he declines the offer, he gets no payment. Moreover, there exist a wide variety of distributions for which we may ignore the restriction that output be monotone in type.

We show that the optimal dynamic mechanism differs sharply from the optimal static one. Even in the case of the uniform distribution, bang-bang procurement is suboptimal until his relational capital is sufficiently high, and so the monotonicity constraint binds for a positive measure of types even for the uniform distribution. Thus, the optimal contract will, at least at low levels of relational capital, involve probabilistic procurement. (Although this result is only stated for the uniform distribution, the underlying logic holds for all distributions.) Consequently, the optimal dynamic contract cannot be implemented via a sequence of take-it-or-leave-it contracts (where the take-it-or-leave-it payment $c_*$ is history dependent).

We establish the existence of a critical level of relational capital $v^*$ such that $v_t < v^*$ implies that the agent is incentivised purely through adjustments to relational capital and $v_t = v^*$ implies that he is a vested partner in the enterprise and is incentivised purely through cash rewards. For $v_t < v^*$ the principal's allocation decision is *ex ante* inefficient and favors the market over the agent. For $v_t = v^*$ the optimal allocation is efficient so that the principal procures from the agent if, and only if, the agent's cost realisation $c_t \leq d$.

The derivative of the principal's value function is shown to be a martingale. Thus, the martingale convergence theorem implies that relational capital process $(v_t)$ will eventually reach the critical absorbing state $v^*$ with probability 1. That is, the agent will eventually become a vested partner under the optimal contract.

We provide an overview of our results in the next subsection, placing them in context, and review related literature in section 1.2. The model is outlined in section 2. In section 3 we formulate the principal's contract design problem as a recursive dynamic program and prove the existence of an optimal contract (ie, an optimal policy). We use techniques from control theory to analyze the optimal contract in section 4. In section 5 we derive both short- and long-run dynamics of the contractual relationship. We provide some brief concluding remarks in section 6. Proofs not in the text appear in the appendix.

### *1.1. Overview of Results*

Our main contribution provides necessary and sufficient conditions for a contract to be optimal. The result of note here is that in contrast to the static setting, bang-bang allocations need not be optimal. This is due to the fact that the principal's value function (which is a function of promised utility) is concave and has unbounded derivative when the agent's promised utility is zero. Therefore, continuation surplus is concave and increasing, and for sufficiently small promised utility, marginal social surplus is unboundedly large. The proof of Proposition 4.2 supposes that a bang-bang allocation is optimal, and then considers lowering continuation utility for the lowest cost type, and raising it for the highest cost types. Because of the bang-bang nature of the posited allocation, a large fraction of high cost types all have increased continuation utility. This raises social surplus, but there is also a loss in social surplus for the low cost types, as well as the concomitant loss in instantaneous surplus that comes with this change in continuation utilities. Once again, because marginal social surplus is unboundedly large for sufficiently small promised utility, we find an improvement to a bang-bang allocation.

We offer another, heuristic, argument that highlights the difference between the static and dynamic settings. In the static setting, incentive compatibility requires that output be nonincreasing in type. This is equivalent to the requirement that the agent's utility function be decreasing and convex as a function of the type $c \in [0, 1]$. Moreover, the derivative of the utility function is precisely the probability of procurement, and hence is bounded below by 0 and above by 1. The space of such utility functions is closed and convex and may be regarded as the principal's control set. The principal's objective is linear on this set of utility functions and hence the optimal control must be an extreme point. A well-known result from convex analysis states that the set of extreme points of this convex set consists of utility functions whose derivatives are either 0 or 1 almost everywhere, so bang-bang procurement is optimal. Notice that this result does not depend on the distribution of types. In the dynamic setting, the principal's control is essentially unchanged. However, the principal's objective function, more specifically, his utility from promising the agent some continuation utility, is concave on the control set, and so extreme points need no longer be optimal. Indeed, as we show by example, even for the uniform distribution, bang-bang allocations are suboptimal for some levels of promised utility (relational capital).

Our findings rely crucially on the dynamic nature of the problem, a feature captured by the continuation utility term. Our next result (Theorem 3) shows that eventually, with probability one, the agent will become a vested partner in the firm. Proposition 5.3 shows that this is a permanent state of affairs in that, upon being vested, the agent remains a vested partner forever. Specifically, in each period, the vested agent gets first right of refusal to produce the good in lieu of a payment of $d$.

It is worth noting that in other studies in dynamic mechanism design, be it in the iid setting of, say, Thomas and Worrall (1990), or the more general (non-iid) setting of, say, Pavan, Segal and Toikka (2014) – both of which are described below – ironing is not a critical feature of the optimal contract, while in our model, ironing cannot be ruled out even for the uniform distribution. The reason for this is that in these other studies, the agent is not faced with the twin constraints of liquidity and participation. For sufficiently small levels of promised utility, these two constraints ensure that the shape of promised utility, as a function of cost type, is very flat, leading to a loss of social surplus. In contrast, in Thomas and Worrall (1990) for instance, there are no lower bounds for the agent's utility, and so it is easy to incentivise the agent by suitably lowering his continuation utility. Heuristically, this leads to a steeper promised utility function. Our results show that liquidity and participation constraints taken together have novel, and heretofore, unexplored effects on the structure of the optimal contract.

### *1.2. Related Literature*

This paper contributes to a growing body of work in dynamic mechanism design. As is common in this line of research, we employ recursive techniques for analyzing dynamic agency problems pioneered by Green (1987) (who studied social insurance), Spear and Srivastava (1987) (who studied dynamic moral hazard), and Thomas and Worrall (1990) (who examined income smoothing under private information), in which shocks are iid over time and the state variable is taken to be the expected present value of the agent's utility under the continuation contract.

The model we study has formal similarities to a strand of research in corporate finance known as dynamic *cash flow diversion* (CFD) models. See, for example, Quadrini (2004), Clementi and Hopenhayn (2006), DeMarzo and Sannikov (2006), DeMarzo and Fishman (2007), and Biais et al. (2007) among others. All of these papers assume a risk-neutral agent who either has limited liability or is less patient than the risk-neutral principal. There are several key differences between the environment we study and the one analyzed in the dynamic CFD literature. First and foremost, the underlying problem facing the principal in CFD models involves moral hazard, ie, a situation in which the agent must be given incentives either not to expropriate privately observed cash flows for his personal use or to privately exert personally costly effort. (As DeMarzo and Fishman (2007) demonstrate, these two situations are formally equivalent.) We, by contrast, study a setting of adverse selection and intra-temporal screening that cannot be interpreted as moral hazard.[2]

Specifically, the principal in our model wishes to tailor her contemporaneous

(2) The conditions under which *ex post* hidden information, as in the CFD models, is analogous to moral hazard are articulated in Milgrom (1987).

policy decision of whether to procure a unit of output in the current period to the agent's private information regarding the realization of his continuously distributed cost of production. Rather than corporate finance, our focus is rooted in questions of procurement and monopolistic screening more readily identified with industrial organization and regulation.[3]

The paper most closely related to this one is Krishna, Lopomo and Taylor (2013). Like us, they study a setting of dynamic procurement in which the agent has iid cost shocks and is liquidity constrained. There are, however, significant technical differences between the two environments studied as well as important differences in results. Formally, Krishna, Lopomo and Taylor (2013) analyze a model in which the principal's utility over output is given by a smooth concave function and the agent's cost realization may take on one of two values. In the current paper, we study a setting of unit demand by the principal and a continuum of cost types for the agent. Hence we analyze a setting that is more granular in one dimension and smoother in another. Studying a continuum of types requires us to use novel optimal control techniques developed by Hellwig (2009). This analysis reveals one of our key insights, namely that in contrast with the static setting, probabilistic procurement of output is generally unavoidable, and that output need not be bang-bang.

There are also several other recent investigations of screening mechanisms in dynamic environments. For instance, Bergemann and Välimäki (2010) introduce and analyze a dynamic version of the VCG pivot mechanism. In a recent article, Pavan, Segal and Toikka (2014) study dynamic screening in a setting in which the distribution of types may be non-stationary and agents' payoffs need not be time-separable. These authors derive a generalization of the envelope formula of Myerson (1981) for incentive compatible static mechanisms and use this to compute a dynamic representation for virtual surplus in the case of quasi-linear preferences. While their analysis is illuminating, the generality of their model prohibits use of the recursive methods at the core of our study.

Boleslavsky and Said (2012) explore a dynamic selling mechanism in which a consumer possesses both permanent private information about his propensity to have high or low taste shocks and transitory private information about his current (conditionally independent) shock. Contrary to our setting, the optimal contract in Boleslavsky and Said's model exhibits allocational inefficiency in the long run – after a sufficiently long time horizon, the supplier will eventually refuse to serve the consumer.

The main difference between the present paper and those mentioned above is that they focus on settings where the agent's information is not iid over time, but

(3) See, for example, Laffont and Martimort (2002, p 86).

where they do not impose strong liquidity constraints. In contrast, our results about ironing of output and non bang-bang output owe their existence to the combination of liquidity and participation constraints.

## 2. The Model

A principal has unit demand for a good in every period $t = 0, 1, 2, \ldots$. Her instantaneous utility from consumption of the good is normalised to 1, while absence of consumption yields her 0. In each period the principal can procure the good from a competitive market at a fixed cost of $d \in (0, 1]$. In addition, the principal has the option of procuring the good from an *agent*, with whom she can enter into a long-term contract. The agent produces the good at a cost of $c \in [0, 1]$, where $c$ is drawn from a distribution $F$ and is iid over time. Moreover, the cost $c$ is privately known to the agent. The distribution $F$ has continuous density $f$, where $f(c) > 0$ for all $c \in [0, 1]$.

Both the principal and the agent discount future utility at the rate $\delta \in (0, 1)$. We assume the principal can *commit* to the contract with the agent, but the agent can leave at any date to an outside option worth 0. Furthermore, we assume that the agent is *liquidity constrained* in every period, and so must be paid at least his cost of production.

We distinguish between a liquidity constraint under which an agent must be paid up front in order to produce and *limited liability* under which an agent cannot be paid a negative wage after producing. Limited liability is often assumed in hidden action models where an agent, though wealth constrained, may exert costly effort to produce output for which he may not be reimbursed, but cannot be further penalized monetarily. In a hidden information setting like ours, wealth of zero precludes agents from producing without being advanced payment by the principal.

The timing runs as follows. At the beginning of the game the principal offers the agent an infinite-horizon contract which he may accept or reject. If he rejects, then the principal procures from the market in every period, while the agent receives a payoff of 0. If the agent accepts the principal's offer, the contract is executed.

## 3. Contract Design

A period-$t$ history $h^t$ is a collection of cost reports in periods $0, 1, \ldots, t-1$. Let $H_t$ denote the set of all period-$t$ histories. (We shall take $H_0 = \{\varnothing\}$.) A contract is a collection of functions $(m_t, q_t) : H_t \times [0, 1] \to [0, 1]^2$, where $m_t(h^t, c_t)$ represents the transfer and $q_t(h^t, c_t)$ is the amount procured from the agent, with both being functions of the past and the current cost report. By the Revelation Principle (eg, Myerson, 1981), the principal may restrict attention to incentive compatible direct mechanisms when designing a contract. Moreover, it is well known (see, eg, Thomas and Worrall, 1990) that in the setting under study, she also may restrict attention to recursive mechanisms in which the state variable is the agent's lifetime promised expected utility under the contract, denoted by $v$. (This latter feature is a consequence of the fact that costs are iid over time.)

Thus, a recursive mechanism is a triple $(m, q, w) : [0, 1] \times \mathbb{R}_+ \to [0, 1] \times [0, 1] \times \mathbb{R}_+$, where for each reported cost $c$ and level of promised utility $v$, a contract specifies the probability of production $q(c, v)$, the instantaneous payment made to the agent $m(c, v)$, and the continuation utility $w(c, v)$, which will serve as the state variable at the beginning of the next period. In what follows, we suppress the dependence on $v$ whenever there is no cause for confusion.

The agent's instantaneous rent is $u(c) := m(c) - cq(c)$. Given this, it is technically convenient to consider contracts of the form $(u, w, q)$ rather than $(m, q, w)$. We now present the contractual constraints under this formulation. Notice that working with $u$ instead of $m$ allows us to formulate the liquidity constraints in a natural way.

**Liquidity:** The agent's liquidity constraints are written as

[Liq] $$u(c) \geq 0 \quad \text{for all } c \in [0, 1]$$

That is, when the agent reports truthfully, the monetary transfer he receives from the principal, $m(c)$, must cover his production costs $cq(c)$. The liquidity constraints do not permit wealth accumulation by the agent because any such saving can be done by the principal on his behalf. In effect, we are assuming that the principal has access to all markets that the agent has access to. Indeed, the principal saves (and dis-saves) on the agent's behalf by adjusting his *relational capital* $v$.

**Promise Keeping:** The ***promise keeping*** constraint that the contract must obey is written as

[PK] $$\int_0^1 \big[u(c) + \delta w(c)\big] f(c)\, \mathrm{d}c = v$$

The agent's lifetime expected payoff $v$ after some history is composed of expectations over his payoff in the current period, $u(c)$, and his continuation payoff, $\delta w(c)$.

**Incentives:** The local incentive compatibility constraint can be written in differential form as

[IC] $$u'(c) + \delta w'(c) = -q(c) \quad \text{for almost all } c \in [0,1]$$

while the global incentive compatibility condition, which is often referred to as an implementability condition and requires that $q$ be monotone decreasing, is stated as

[Mon] $$q'(c) \le 0 \quad \text{for all } c \in [0,1]$$

with the understanding that at points where the derivative of $q$ is not well defined, we shall take $q'(c) \le 0$ to mean $\limsup_{\varepsilon \searrow 0} \big[q(c+\varepsilon) - q(c-\varepsilon)\big]/\varepsilon \le 0$. Thus, at any point of discontinuity, this 'derivative' is $-\infty$. The contract $(u, w, q)$ is incentive compatible if, and only if the probability of production $q$ is (weakly) decreasing in $c$, and if the utility of type $c$ is given by (integrating) [IC], ie, if $u(c) + \delta w(c) = u(1) + \delta w(1) + \int_c^1 q(s)\,\mathrm{d}s$.

**Participation:** The agent has an outside option of 0 that he can exercise at any point in time. The continuation utility $w(c)$ is the sum of expected future rents, and because instantaneous rents to the agent can never be negative, it follows that we must include feasibility constraints that require $w(c) \ge 0$ for all $c \in [0,1]$. Thus, promise keeping [PK] implies that the agent's lifetime expected utility $v$ is always nonnegative, and the participation constraint that the contract initially offer him nonnegative lifetime utility may be ignored.

The following proposition shows that the principal's problem can be written as a dynamic program, and establishes that an optimal contract exists by virtue of being the corresponding policy function.

**Theorem 1.** *The principal's discounted expected utility under an optimal contract,* $(u, q, w)$*, is represented by a unique, concave, and continuously differentiable function* $P : \mathbb{R}_+ \to \mathbb{R}$ *that satisfies*

[VF]
$$P(v) = \max_{(u,q,w)} \int_0^1 \big[(1-d) + (d-c)q(c) + \delta\big(P(w(c)) + w(c)\big)\big] f(c)\,\mathrm{d}c - v$$

*subject to: promise keeping* [PK]*, the incentive compatibility conditions* [IC] *and* [Mon]*, liquidity* [Liq]*, and feasibility* $q(c) \ge 0$ *and* $w(c) \ge 0$ *for all* $c \in [0,1]$*. Moreover,*

(a) $P(0) = (1-d)/(1-\delta)$,

(b) *there exists* $v^* \in [0, \infty)$ *such that* $P'(v) > -1$ *for* $0 \le v < v^*$ *and* $P'(v) = -1$ *for* $v \ge v^*$*, where* $v^* := \int_0^d F(s)\,\mathrm{d}s/(1-\delta)$*, and*

(c) $P'(0) = \infty$.

Notice that social surplus $P(v) + v$ is maximised at any $v \ge v^*$. In particular, social surplus is an increasing function of $v$, that grows until $v = v^*$. The level of relational capital $v^*$ is easily interpreted. If the principal procures from the agent whenever $c \le d$, and makes a payment of $d$, then the agent's instantaneous expected utility is $\int_0^d F(s)\,\mathrm{d}s$. Therefore, $v^*$ is the lifetime utility that accrues to the agent if the principal always gives the agent the right of first refusal to produce the good for a payment of $d$. The following result shows that compensation in any optimal contract must be backloaded.

**Proposition 3.1.** For any optimal contract $(u, w, q)$, payments are backloaded, ie, for any $v \ge 0$,

$$w(c, v) < v^* \text{ implies } u(c, v) = 0 \text{ for almost all } c \in [0, 1]$$

*Proof.* Suppose, by way of contradiction, that there is an optimal contract $(u, w, q)$ where both $w(c) > 0$ and $u(c) > 0$ for some $v \in [0, v^*)$ and for all $c \in \Lambda \subset [0, 1]$, where $\Lambda$ has positive (Lebesgue) measure. Then, consider the alternative contract $(\tilde{u}, \tilde{w}, q)$ where $\tilde{u}(c, v) = 0$ and $\delta\tilde{w}(c, v) = u(c, v) + \delta w(c, v)$. Notice that the new contract $(\tilde{u}, \tilde{w}, q)$ does not violate any of the constraints, and gives the agent the same utility as the original contract. The term $u(c)$ does not enter the principal's value function, while $P(w) + w$ is *strictly* increasing on the set $[0, v^*]$. Because the set $\Lambda$ has positive measure, the principal does strictly better with the new contract $(\tilde{u}, \tilde{w}, q)$, which is impossible given the optimality of the original contract. □

The intuition behind this result is that in the dynamic setting, the principal can induce truth telling via two instruments: instantaneous rent $u(c)$ and continuation utility $w(c)$, the latter being the sum of expected future rents. The problem with providing incentives through current rent, $u(c)$, is that this must be non-negative due to the liquidity constraints; thus, the agent can only be rewarded and never penalized. Moreover, any instantaneous rent awarded to the agent is spent outside the contractual relationship and therefore does not benefit the principal. If, however, the principal chooses to provide the necessary incentives through continuation payoffs $w(c)$, then she can reward the agent by adjusting his relational capital up or penalize him by adjusting it down. Hence, providing incentives through continuation utility has two advantages: it keeps payments inside the relationship and it permits penalties. Once $v = v^*$, liquidity constraints no longer bind (ie, penalties become irrelevant), and the principal can provide the requisite incentives purely through instantaneous rents.

In light of proposition 3.1, we define the class of ***maximal rent*** (optimal) contracts wherein $w(c, v) \leq v^*$ for all $c \in [0, 1]$, and $w(c, v) < v^*$ implies $u(c, v) = 0$. In such a contract, payments are deferred until the agent's promised utility reaches $v^*$, after which the agent is compensated in each period for production, and gets paid his production cost plus the associated information rent. Intuitively, in a maximal rent contract, the agent gets paid as soon as possible.

## 4. Optimal Contracts

In the previous section, we described the existence of a value function, and characterized basic properties of optimal contracts. To gain a further understanding of the procurement decision in a given period, we next formulate the principal's instantaneous problem as an optimal control problem. Recall that by [IC], $u(c) + \delta w(c) = u(1) + \delta w(1) + \int_c^1 q(s)\,\mathrm{d}s$, while promise keeping [PK] requires that $\int_0^1 \big[u(s) + \delta w(s)\big] f(s)\,\mathrm{d}s = v$. Substituting $u(c) + \delta w(c)$ in [PK] gives us $u(1) + \delta w(1) + \int_0^1 \int_c^1 q(s) f(c)\,\mathrm{d}s\,\mathrm{d}c = v$. Upon integrating by parts, we see that $\int_0^1 \int_c^1 q(s) f(c)\,\mathrm{d}s\,\mathrm{d}c = \int_0^1 q(s) F(s)\,\mathrm{d}s$, which allows us to rewrite [PK] as

$$\text{[PK*]} \qquad -\big(u(1) + \delta w(1)\big) + \int_0^1 \big(v - q(c)F(c)\big)\,\mathrm{d}c = 0$$

Notice that [IC] and [PK*] imply [PK]. This is because once $q$ is specified, $u(c)+\delta w(c)$ is determined, up to a constant of integration, by $q$ as in [IC]. Both [PK] and [PK*] are equivalent ways of determining the constant of integration. We are now ready to examine the allocation problem faced by the principal in each period.

### *4.1. Optimal Allocations – A Relaxed Look*

Consider the principal's allocation problem after a history wherein the agent is owed $v$ utiles. We begin by examining the *relaxed* problem, in which we ignore the monotonicity constraint [Mon].

With this in mind, the principal's problem is

$$\max_{(u,q,w)} \big[(1 - d) + (d - c)q + \delta\big(P(w) + w\big)\big] f(c) - v$$

subject to [IC] and [PK*]. (The maximum value, which is attained in this setting, is at least as large as $P(v)$.) Notice that in light of proposition 3.1, rents will only be paid if $w(c) = v^*$. Therefore, there is another optimal contract that has $u(c) = 0$ for all $c$, which clearly implies $w(c)$ may exceed $v^*$. This allows us to ignore the liquidity constraint [Liq]. In light of this, we may write the Hamiltonian for the problem as

$$\begin{aligned} H(c, q(c), w(c)) = \big[1 - d + (d - c)q + \delta\big(P(w) + w\big)\big] f(c) - v \\ - \lambda(c)q(c) + \eta\big[v - q(c)F(c)\big] \end{aligned}$$

where $q(c)$ is the control variable, $w(c)$ is the state variable, [IC] describes the evolution of the state, $\lambda(c)$ is the costate variable for [IC], and $\eta$ is the multiplier for [PK*]. The adjoint equation for this problem requires that $\lambda'(c) = -H_w$, which implies

[Adj-r] $$\lambda'(c) = -\delta\big[P'(w(c)) + 1\big] f(c)$$

Because $H$ is linear in $q$, it follows that the optimal choice of $q$ is given as follows:

[Opt-r] $$q(c) \begin{cases} = 1 & \text{if } d > c + \frac{\lambda(c)}{f(c)} + \eta\frac{F(c)}{f(c)} \\ \in [0, 1] & \text{if } d = c + \frac{\lambda(c)}{f(c)} + \eta\frac{F(c)}{f(c)} \\ = 0 & \text{if } d < c + \frac{\lambda(c)}{f(c)} + \eta\frac{F(c)}{f(c)} \end{cases}$$

The value of $w(0)$ is unspecified, so the transversality condition requires that $\lambda(0) = 0$. Of course, $w(1)$ is specified according to [PK*], so it follows from a Theorem of Hestenes—see, for instance, Takayama (1985, Theorem 8.C.4, p. 658)—that

$$\lambda(1) = -\delta\eta$$

This allows us to integrate $\lambda'(c)$ to obtain

$$\lambda(c) = -\delta \int_0^c \big[P'(w(s)) + 1\big] f(s)\, \mathrm{d}s$$

Noting that $\lambda(1) = -\delta\eta$, we find

[4.1] $$\eta = \int_0^1 \big[P'(w(s)) + 1\big] f(s)\, \mathrm{d}s$$

Let us define

[4.2] $$\psi(c, v) := c + \eta\frac{F(c)}{f(c)} - \delta\left[\frac{\int_0^c \big[P'(w(s)) + 1\big] f(s)\, \mathrm{d}s}{f(c)}\right]$$

to denote the agent's *virtual cost* associated with cost $c$. The terms $c + \eta F(c)/f(c)$ correspond to the static part of the virtual cost, while the last term

$$-\delta\left[\int_0^c \big[P'(w(s)) + 1\big] f(s)\, \mathrm{d}s\right] / f(c)$$

corresponds to the dynamic part of the virtual cost.

It is easy to see that the optimal choice of $q(c)$ can now be formulated as

[Opt-r]
$$q(c) \begin{cases} = 1 & \text{if } d > \psi(c, v) \\ \in [0, 1] & \text{if } d = \psi(c, v) \\ = 0 & \text{if } d < \psi(c, v) \end{cases}$$

The special case of $\delta = 0$ gives us precisely the virtual cost function for the static problem with a promise keeping constraint (see Appendix A). Just as in the static problem with a promise keeping constraint, the virtual cost function $\psi(c, v)$ depends on $\eta$, which in turn depends on $v$, but unlike the static problem, the virtual cost also depends on the future marginal social surplus of all superior types. We record some facts about the virtual cost function.

**Proposition 4.1.** The virtual cost function $\psi(c, v)$ is continuous in $c$. For each $v < v^*$, $\psi(d, v) > d$, so that a positive measure of types is always excluded.

While it is desirable to always exclude some types, it is still not clear what the form of the optimal procurement schedule looks like. In the static model with promise keeping, the standard regularity assumption that $F(c)/f(c)$ is non-decreasing is sufficient to ensure that the virtual cost function is increasing, implying that monotonicity constraints never bind. Indeed, if $F(c)/f(c)$ is non-decreasing, then the static virtual cost function $c + \eta F(c)/f(c)$ is strictly increasing in $c$ (for every fixed level of $v$), which implies that the optimal policy will necessarily be bang-bang.

By contrast, in the dynamic setting, even if $F(c)/f(c)$ is non-decreasing, it may well be the case that $\psi(c, v)$ is non-monotone in $c$. In such a case, ironing will be necessary. The next proposition shows that even in the case where $F$ is uniform, for sufficiently small $v$, bang-bang allocations are suboptimal. Thus, there exists a positive measure of types for which $\psi(c, v) = d$.

**Proposition 4.2.** Suppose costs are uniformly distributed, so $F(c) = c$. Then, for all $v$ sufficiently small, there exists a set $C_0$ of positive measure of costs such that for all $c \in C_0$, $\psi(c, v) = d$.

This result stands in stark contrast to the static model, because it says that the monotonicity (or implementability) condition will bind for sufficiently small levels of $v$. To see the intuition for why a bang-bang allocation may not be optimal, suppose $v$ is small and suppose (by way of contradiction, that) a bang-bang allocation is optimal. Taking $v$ small ensures that $u = 0$. By considering a $q$ that is a step function but not bang-bang, we raise $w(d)$, which has the effect of reducing $w(0)$. The change in $q$ entails some instantaneous loss of social surplus, but has the effect of increasing continuation social surplus, and the change in instantaneous social surplus is linear at the margin. Because $v$ is small, $w(d)$ is small, and because $P'(0) = \infty$, the marginal increase in continuation social surplus is very large because a large fraction of cost types benefits from this improvement, and thus dominates the instantaneous loss in social surplus, ensuring that we can improve on the bang-bang allocation.

This suggests that we should formulate the optimal control problem in such a way that it is cognizant of the monotonicity constraint [Mon]. A standard approach following Guesnerie and Laffont (1984) takes the allocation $q$ as a state variable, and requires its derivative, $q'(c) = z(c)$ to be a control variable, with the monotonicity condition being reflected by the requirement that $z(c) \leq 0$ for all $c$. Unfortunately, standard optimal control techniques, even those requiring minimal hypotheses—see, for instance, Clarke (1983)—require the state variable $q$ to be absolutely continuous. This rules out, among other things, discontinuous allocations, and it is a priori not clear if $q$ must necessarily be absolutely continuous. (Recall that $q$ in the static setting is not continuous. Then, it is not clear what the derivative of $q$ is at points of discontinuity.) Nevertheless, Hellwig (2009) has reformulated the maximum principle of Clarke so as to accommodate monotonicity constraints, without requiring the allocation to be absolutely continuous. We, therefore, consider the problem of finding an optimal allocation using this version of the maximum principle in the next section.

### *4.2. Optimal Allocations – A Less Relaxed look*

As before, the problem of optimal allocation requires the principal to solve

[P] $$\max_{(u,w,q)} \big[(1-d) + (d-c)q(c) + \delta\big(P(w(c)) + w(c)\big)\big] f(c) - v$$

subject to [IC], [PK], and [Mon] and feasibility. To formulate this problem as an optimal control problem, we follow Hellwig (2009). Notice that because $q$ is monotone decreasing, it can be represented as the sum of an absolutely continuous function and a singular function. (This is just the Lebesgue decomposition of $q$.) Let $z(c)$ be the Radon-Nikodym derivative of the absolutely continuous part of $q$. This will be our control variable. The state variables will be $q(c)$ and $w(c)$. As in the relaxed approach above in section 4.1, we shall consider contracts where $u = 0$ for all $c$. The evolution of $w$ is governed by the state equation [IC]; let $\lambda(c)$ be the costate variable for this equation. The integral constraint [PK] has the (constant) Lagrange multiplier $\eta$. The monotonicity constraint [Mon] is captured by the requirement that

$-z(c) \geq 0$; let $\xi(c)$ be the multiplier for this constraint. There are two feasibility constraints to consider:

$$q(c) \geq 0 \qquad \text{[Feas1]}$$

$$q(c) = 1 \qquad \text{[Feas0]}$$

Let $\beta(c)$ be the multiplier for [Feas1] and $\gamma$ the (constant) multiplier for [Feas0]. We are now ready to state the Hamiltonian.

$$H(\cdot) = \big[(1-d) + (d-c)q(c) + \delta\big(P(w(c)) + w(c)\big)\big]f(c) - v$$
$$\lambda(c)(-q(c)) + \delta\eta\big[v - w(c)f(c)\big] - \xi(c)z(c) + \beta(c)q(c)$$

The adjoint equations are given by

$$\lambda'(c) = -H_w = -\delta\big[P'(w(c)) + 1\big]f(c) + \delta\eta f(c) \qquad \text{[Adj-w]}$$

and

$$\xi'(c) = -H_q = -(d-c)f(c) + \lambda(c) - \beta(c) \qquad \text{[Adj-q]}$$

Notice that there are no boundary conditions for $w$, since the level of $w$ will be determined by the promise keeping constraint. Therefore, $\lambda(1) = \lambda(0) = 0$. Integrating, we find

$$\lambda(c) = -\delta \int_0^c \big[P'(w(c)) + 1\big]f(c)\,\mathrm{d}c + \eta\delta F(c)$$

In particular, the fact that $\lambda(1) = 0$ implies

$$-1 + \eta = \int_0^1 P'\big(w(c)\big)f(c)\,\mathrm{d}c \qquad \text{[4.3]}$$

By theorem 3.1 of Hellwig (2009), we obtain $\xi \leq 0$ and $\xi(1) = 0$. Therefore, we can integrate [Adj-q] to obtain

$$\xi(c) = \int_c^1 \big[(d-c)f(c) - \lambda(c) + \beta(c)\big]\,\mathrm{d}c \qquad \text{[4.4]}$$

By Theorem 3.1 of Hellwig (2009), a necessary and sufficient condition for optimality with respect to $z \leq 0$ is $\xi \leq 0$ and $\xi z = 0$.

Note the rôle of $\beta(c)$ here. In the static case, if $c \geq d$, the socially optimal solution calls for $q = -\infty$ if we are allowed to ignore [Feas0]. At such cost levels, $\beta$ is positive. However, as noted in proposition 4.2, in the dynamic setting (where $\delta > 0$), it no longer follows that bang-bang allocation policies are optimal. We summarise our discussion here with

**Theorem 2.** *The optimal policy $q(c, v)$ corresponds to maximising the Hamiltonian by choosing $\xi$ such that $\xi \leq 0$ and $\xi z = 0$.*

In conjunction, proposition 4.2 and Theorem 2 say that due to liquidity constraints, the optimal allocation in the dynamic setting is very different from the static setting. The static setting (even in the presence of promise keeping) entails bang-bang allocations, and for suitable distributions of types, we may ignore the monotonicity constraint [Mon]. In contrast, in the dynamic setting, we cannot ignore the monotonicity constraint [Mon], and it may well be that bang-bang allocations are suboptimal.

## 5. Dynamics

Finally we derive both short- and long-run dynamics of the contractual relationship. It is easy to see from the envelope condition for the Hamiltonian that $P'(v) + 1 = \eta$. We thus obtain

**Lemma 5.1.** An optimal contract induces a process $P'$ that is a martingale: ie,

$$P'(v) = \int_0^1 P'\big(w(c)\big) f(c)\,\mathrm{d}c$$

*Proof.* The envelope condition states that $P'(v) = -1 + \eta$, while we have established in [4.3] above that $-1 + \eta = \int_0^1 P'\big(w(c)\big) f(c)\,\mathrm{d}c$, which concludes the proof. □

**Proposition 5.2.** In an optimal contract, for all $v \in (0, v^*)$, we have $P'(w(1)) > P'(v) > P'(w(0))$. Moreover, $w(0, v) > v > w(1, v)$.

By [PK*], we have $u(1) + \delta w(1) = v - \int_0^1 q(c)F(c)\,\mathrm{d}c$. This implies $w(0, v) > v > w(1, v)$ for all $v \in (0, v^*)$. In other words, incentive compatibility and promise keeping force the principal to spread out the agent's continuation utilities, rewarding him for favorable (low) cost reports and penalizing him for unfavorable (high) cost ones. The proposition shows that, in fact, $P'(w(1)) > P'(v) > P'(w(0))$. This follows easily if $P$ is strictly concave. The proposition shows this to be the case even though we are unable to establish that $P$ is strictly concave.

We are now in a position to describe the long-run properties of the optimal contract.

**Theorem 3.** *In any optimal contract, the principal eventually (with probability one) procures from the agent if, and only if, the cost realisation in any period is less than $d$. In other words, eventually, the principal will have no distortions in procurement (though she will still pay the corresponding information rents). Formally, we have that the martingale $P'$ converges almost surely to $P'_\infty = -1$.*

From the martingale convergence theorem, it follows that $P'$ must converge, almost surely, to an integrable random variable $P'_\infty$. The theorem establishes that along almost all sample paths, this limit must be $-1$. That $P'$ cannot settle down to a finite limit greater than $-1$ follows from proposition 5.2 above and the continuity of the contract in $v$.

Proposition 5.2 says that for $v \in (0, v^*)$, the agent is rewarded for reporting a low cost realization and penalized for reporting a high one. Because $P'(0) = \infty$, an arbitrarily long string of penalties never pushes the agent's continuation utility into the absorbing state at $v = 0$. An arbitrarily long string of rewards, however, will eventually drive his continuation utility into the absorbing state at $v = v^*$. It is possible to show that with positive probability, the martingale $P'$ can take all values in $(-1, \infty)$. Moreover, we can show that for any starting value $v \in (0, v^*)$, the martingale $P'$ reaches $v^*$ in finite time with probability one. Proofs of these statements are omitted because they are very similar to propositions in Krishna, Lopomo and Taylor (2013).

Theorem 3 says that with probability 1, the agent will eventually experience a sufficiently long sequence of rewards to become a vested partner in the firm. In fact, the optimal maximal rent contract at $v = v^*$ is particularly easy to characterize.

**Proposition 5.3.** Upon becoming a vested partner, in a maximal rent contract there is no longer any output distortion so $q(c) = 1$ if $c \le d$ and 0 otherwise, and promised utility $w(c) = v^*$ for all $c \in [0, 1]$. Furthermore, this contract is implemented by giving the inside agent, in each period, a first right of refusal for production in exchange for a fixed payment of $d$.

*Proof.* The *proof* of part (b) of Theorem 1 implies that for $v = v^*$, we must have no distortion in output, ie, $q(c) = 1$ if $c \le d$ and 0 otherwise, and promised utility $w(c) = v^*$ for all $c \in [0, 1]$.

This results in a stream of non-negative rents according to $u(c) := u(1) + \int_c^{c \vee d} q(x)\, \mathrm{d}x$, where $u(1) = 0$. Then, $u(c) = \max[d - c, 0] = 0$ for all $c \ge d$, and is equal to $d - c$ for all $c \le d$. Therefore, the *expected* rent in any period is $\int_0^1 u(c) f(c)\, \mathrm{d}c = \int_0^d (d - c) f(c)\, \mathrm{d}c = F(d)\big(d - \mathbf{E}[c \mid c \le d]\big)$. Since the agent will receive, upon becoming a vested partner, precisely this expected utility in every period, it follows that

$$v^* := \frac{F(d)\big(d - \mathbf{E}[c \mid c \le d]\big)}{1 - \delta}$$

This can be implemented as follows: The principal offers the fully vested agent the option to produce in exchange for a payment of $d$. Clearly, the agent will only accept if his true cost is less than $d$, an event whose probability is $F(d)$. The agent's expected cost of production is precisely $\mathbf{E}[c \mid c \leq d]$, which shows why $v^*$ is a vested agent's lifetime expected utility. □

The principal's utility at this stage of the relationship is then given by equation $P(v^*)+v^* = \max_{(u,w,q)} \int_0^1 \left[1-d+(d-c)q(c)\right] f(c)\, \mathrm{d}c + \delta\big(P(v^*)+v^*\big)$. As noted above, with $v = v^*$, there are no more output distortions, so $q(c) = 1$ if $c \leq d$ and 0 otherwise. Therefore, $\big(P(v^*) + v^*\big)(1-\delta) = (1-d) + \int_0^d (d-c) f(c)\, \mathrm{d}c = (1-d) + F(d)\big(d - \mathbf{E}[c \mid c \leq d]\big) = (1-d) + v^*(1-\delta)$.

This implies that the principal's expected utility when the agent is fully vested is

$$P(v^*) = \frac{1-d}{1-\delta}$$

so that the principal pays $d$ in every period, regardless of whether she procures from the fully vested agent or from a competitive market. Notice that $P(v^*) = P(0)$, meaning that eventually, all the social surplus generated by producing at a cost less than $d$ goes to the agent, with the principal being exactly indifferent to the presence of the agent. However, this is a consequence of all rents being backloaded – from the moment the agent was hired at some initial level of promised utility (which is easily seen to be strictly greater than 0), the agent's payments have been deferred until he becomes a vested partner, and all social surplus up to that point has been captured by the principal.

## 6. Conclusion

In this paper we analyze a setting in which a principal wishes to procure a single unit of a good in each period over an infinite horizon. She may procure from the market at known cost $d$ or from an agent who has privately known cost drawn each period from a continuous distribution. The agent is risk-neutral but liquidity constrained and so must be advanced his reported cost by the principal in every period when he produces.

We formulate the resulting dynamic mechanism design problem as a recursive program in which the agent's promised utility at any point is the relevant state variable that is naturally interpreted as his *relational capital*. We prove two results that contrast sharply with the solution to the static problem. First, ironing of the output cannot be ruled out by any regularity conditions on the distribution of costs. Second, the optimal allocation is not generally characterized by a cost cutoff, ie, it will involve probabilistic procurement for some levels of relational capital (promised utility).

We prove that the agent eventually builds up enough relational capital to become a vested partner in the enterprise. Prior to attaining this state, the agent is incentivised exclusively through adjustments to his relational capital, while after achieving it he receives non-negative rents paid in cash.

Several avenues remain open for future work. For example, it would be interesting to study the impact of liquidity constraints under more general stochastic processes governing the agent's cost. Also, enriching the model by considering a setting with multiple agents would be an edifying direction for further investigation.

# Appendices

## A. The Static Model

In this section, we consider the static problem with a promise keeping constraint. The problem is as stated in the text, with the additional resrtiction that $\delta = 0$.

In that case, the incentive constraint reduces to

[IC-static] $$u'(c) = -q(c) \quad \text{for almost all } c \in [0, 1]$$

For the moment, we will ignore the implementability constraint [Mon]. The promise keeping constraint reduces to

[PK-static] $$\int_0^1 u(c) f(c) \, \mathrm{d}c = v$$

We shall formulate this as an optimal control problem with $q$ as the control variable, and $u'$ as the state variable. The (static) value function can be written as $P_0(v)$, where

[VF-static] $$P_0(v) = \max_{(u,q)} \int_0^1 \big[(1-d) + (d-c)q(c)\big] f(c) \, \mathrm{d}c - v$$

subject to [IC-static] and [PK-static]. In the formulation of $P_0(V)$, notice that the instantaneous utility to the principal is $(1-d) + (d-c)q(c) - u(c)$. Using [PK-static] results in $P_0(v)$ above in [VF-static].

The condition [IC-static] governs the evolution of the state variable, and we shall let $\lambda(c)$ denote the costate variable for this equation. Similarly, we shall let $\eta$

denote the multiplier for the constraint [PK-static]. The (static) Hamiltonian can be written as

$$H_0\big(c, q(c), u(c), \lambda(c), \eta, v\big) = \big[(1-d)+(d-c)q(c)\big]f(c) - \lambda(c)q(c) + \eta\big[v - u(c)f(c)\big]$$

The optimality condition for $q$, namely $H_{0,q} = 0$, results in

[FOC-q] $$d \geq c + \lambda/f(c)$$

while the adjoint equation is

[Adj-u] $$\lambda'(c) = -\frac{\partial H_0}{\partial u} = \eta f(c)$$

Because $u(0)$ is free, $\lambda(0) = 0$, so integrating [Adj-u], we get $\lambda(c) = \eta F(c)$. Substituting in [FOC-q], we see that principal procures from the inside agent (with probability 1) if, and only if,

$$c + \eta\frac{F(c)}{f(c)} \leq d$$

The envelope condition is $P_0'(v) = -1 + \eta$. Notice that the optimal choice of $v$ is such that $\eta = 1$, and this results in the standard procurement rule, where the principal procures from the agent if, and only if, $c + F(c)/f(c) \leq d$.

Thus far, we have said nothing of the implementability condition [Mon]. If we make the assumption that $F(c)/f(c)$ is decreasing in $c$ – a property known as *decreasing inverse hazard rate* – the implementability condition [Mon] is immediately satisfied. Of course, if this condition does not hold, then one has to *iron* out the non-monotonicities in the optimal $q$.

It is easy to verify that $P_0(0) = 1 - d$, and that $P_0$ is concave. We shall now establish that the derivative of $P_0$ at 0 is unbounded above.

**Lemma A.1.** The function $P_0(v)$ is continuous, and has $P_0'(0) = \infty$.

*Proof.* The continuity of $P_0$ is easy to establish, and so we omit it here. To see the derivative, notice that integrating [IC-static], we obtain $u(c) = u(1) + \int_c^1 q(x)\,\mathrm{d}x$. Using this in [PK-static], we obtain $v - u(1) = \int_0^1 q(x)F(x)\,\mathrm{d}x$. For $v$ sufficiently small, there exist contracts where we may set $u(1) = 0$, without violating [PK-static]. Therefore, consider a contract where $u(1) = 0$, $\bar{c}(v)$ is such that $v =: \int_0^{\bar{c}(v)} F(x)\,\mathrm{d}x$, and $q(c) = 1$ if, and only if, $c \leq \bar{c}(v)$. Define $u(c)$ by $u(c) = \int_c^1 q(x)\,\mathrm{d}x$. By construction, this contract is incentive compatible, and satisfies promise keeping.

The requirement $v =: \int_0^{\bar{c}(v)} F(x)\,\mathrm{d}x$ implicitly defines the function $\bar{c}(v)$, and has the property that $\bar{c}(v) \searrow 0$ as $v \searrow 0$. Moreover, differentiating both sides of the expression with respect to $v$ gives us $1 = F(\bar{c}(v))\bar{c}'(v)$.

The utility to the principal from this contract is denoted by $\varphi(v) := \int_0^{\bar{c}(v)}(1-c)f(c)\,\mathrm{d}c + \int_{\bar{c}(v)}^1 (1-d)f(c)\,\mathrm{d}c - v = (d-c)F\big(\bar{c}(v)\big) + (1-d) + \int_0^{\bar{c}(v)} F(c)\,\mathrm{d}c - v$. Then, $\varphi'(v) = (d-c)f\big(\bar{c}(v)\big)/F\big(\bar{c}(v)\big) - 1$, where we have used the fact that $1 = F(\bar{c}(v))\bar{c}'(v)$. This implies $\lim_{v\to 0}\varphi'(v) = \infty$, because $f(0) > 0$. Observe that $\varphi(0) = P_0(0)$, although $\varphi(v) \leq P_0(v)$ for all $v \geq 0$.

Finally, note that $P_0(v)$ is concave, so that for all $v \geq 0$, $P_0'(0) \geq \big[P_0(v) - P_0(0)\big]/v$, so that

$$\begin{aligned} P_0'(0) &= \lim_{v\searrow 0} \frac{P_0(v) - P_0(0)}{v} \\ &\geq \lim_{v\searrow 0} \frac{\varphi(v) - \varphi(0)}{v} \\ &= \varphi'(0) = \infty \end{aligned}$$

which completes the proof. □

## B. Proofs from Section 3

We begin with a proof of Theorem 1.

*Proof of Theorem 1.* The proof is somewhat non-standard, because for each $v \in [0,\infty)$, the control set of permitted $(u,w,q)$ is non-compact because the control set is infinite dimensional. Therefore, we resort to some indirect methods to establish the desired properties. Nevertheless, some parts of the proof borrow from the proof of existence in Krishna, Lopomo and Taylor (2013).

We shall begin by showing that there exists a function $P : \mathbb{R}_+ \to \mathbb{R}$ that satisfies

[VF-sup]

$$P(v) = \sup_{(u,q,w)} \int_0^1 \big[(1-d) + (d-c)q(c) + \delta\big(P(w(c)) + w(c)\big)\big]f(c)\,\mathrm{d}c - v$$

subject to feasibility, promise keeping [PK], the incentive compatibility conditions [IC] and [Mon], and liquidity [Liq]. After establishing the existence of such a $P$, we shall show that the supremum is actually achieved for each $v$, which gives us the value function [VF].

First, as a lower bound for $P$, notice that the principal can always just give the agent $v$ utiles without requiring any production. This would give the agent $v$ utiles and cost the principal $-v$ utiles, thus forming a lower bound for her utility.

An upper bound for the principal's value function obtains if we consider the case where there is full information, in which case, the principal's utility is

$$\frac{1}{1-\delta}\int_0^1 \max[1-c, 1-d]f(c)\,\mathrm{d}c - v$$

This entails giving the inside agent exactly $v$ utiles (net of production costs), but paying the true cost of production in each period. Therefore, the value function $P(v)$ must lie within these bounds, ie, must satisfy

$$0 \;\le\; P(v) + v \;\le\; \frac{1}{1-\delta}\int_0^1 \max[1-c, 1-d]f(c)\,\mathrm{d}c$$

Let $\mathbb{R}^{[0,\infty)}$ be the space of all real functions on $[0,\infty)$, and let

$$\mathcal{F} := \left\{Q \in \mathbb{R}^{[0,\infty)} : 0 \le Q(v) + v \le \frac{1}{1-\delta}\int_0^1 \max[1-c, 1-d]f(c)\,\mathrm{d}c\right\}$$

be endowed with the 'sup' metric, which makes it a complete metric space. (It is easy to see that $\mathcal{F}$ is isomorphic to a closed subset of the normed space $\mathbf{C}_b[0,\infty)$ of all continuous and bounded functions on $[0,\infty)$. Indeed, let $\Phi(Q)(v) := Q(v) + v \in \mathbf{C}_b[0,\infty)$, so that $\Phi$ is an isometric continuous bijection.)

Let $\mathcal{F}_1$ be the set of all concave functions in $\mathcal{F}$, let $\mathcal{F}_2$ be the set of all functions $Q \in \mathcal{F}$ such that $Q(v)+v$ is constant for all $v \ge v^*$, where $v^* := \int_0^d F(x)\,\mathrm{d}x/(1-\delta)$, and let $\mathcal{F}_3$ be the set of all functions in $\mathcal{F}$ that are continuous at 0. It is easy to see that $\mathcal{F}_i$ is a closed subset of $\mathcal{F}$ for $i = 1, 2, 3$.

Let $\Gamma(v) := \{(u,q,w) \in (\mathbb{R}^{[0,1]})^3\}$ such that $(u,q,w)$ satisfies [PK], [IC], [Mon], [Liq], and the feasibility constraints. Thus, $\Gamma(v)$ represents the set of permissible $(u,q,w)$.

Define the operator $\mathsf{T} : \mathcal{F} \to \mathcal{F}$ as

$$\begin{aligned}(\mathsf{T}Q)(v) = \sup_{(u,q,w)} \int_0^1 \big[&(1-d) + (d-c)q(c) \\ &+ \delta\big(Q(w(c)) + w(c)\big)\big]f(c)\,\mathrm{d}c - v \\ \text{s.t.}\;\; &(u,q,w) \in \Gamma(v)\end{aligned}$$

for each $Q \in \mathcal{F}$. It is easy to see that $\mathsf{T}Q(v) + v \ge (1-d)/(1-\delta)$ for all $v \ge 0$. Similarly, because $Q(v) + v \le \frac{1}{1-\delta}\int_0^1 \max[1-c, 1-d]f(c)\,\mathrm{d}c$, it follows that $\mathsf{T}Q(v) + v \le \frac{1}{1-\delta}\int_0^1 \max[1-c, 1-d]f(c)\,\mathrm{d}c$, so $\mathsf{T}$ is a well defined operator.

Consider first the case where $Q \in \mathcal{F}_1$, and notice that $\Gamma(v)$ is convex for each $v \ge 0$. Moreover, if $v, v' \ge 0$, $(u,q,w) \in \Gamma(v)$, and $(u',q',w') \in \Gamma(v')$, then for all $\alpha \in [0,1]$, $\alpha(u,q,w) + (1-\alpha)(u',q',w') \in \Gamma\big(\alpha v + (1-\alpha)v'\big)$. This is because all the constraints that define $\Gamma(v)$ are linear inequalities. We can now adapt the arguments in Stokey, Lucas and Prescott (1989, Theorem 4.8, p 81) to conclude that if $Q \in \mathscr{F}_1$, we must also have $\mathsf{T}Q$ concave.

Let us now assume that $Q \in \mathscr{F}_2$ so that $Q'(v) = -1$ for all $v \geq v^*$. Consider the relaxed problem

$$\sup_{(u,q,w)} \int_0^1 \big[(1-d) + (d-c)q(c) + \delta\big(Q(w(c)) + w(c)\big)\big] f(c)\,\mathrm{d}c - v$$
$$\text{s.t. [PK] and Feasibility}$$

where $v \geq v^*$. It is easy to see that every *feasible* solution to this problem must have $q(c) = 1$ for all $c \in [0, d]$ and $q(c) = 0$ otherwise. With this choice of $q$, any choice of $u(c) + \delta w(c)$ that satisfies [PK] with the property that $w(c) \geq v^*$ cannot be improved upon, because $Q(w) + w$ is a constant for $v \geq v^*$. So, let us set $w(c) = v^*$ for all $c \in [0, 1]$, set $u(1) = v - \delta v^* - \int_0^d F(c)\,\mathrm{d}c$, and $u(c) := u(1) + \int_c^1 q(x)\,\mathrm{d}x = u(1) + \max[d - c, 0]$. This choice of $(u, w, q)$ is a solution to the relaxed problem. But by construction, $(u, w, q)$ also satisfies [IC] and [Mon]. (Indeed, our choice of $u(1)$ comes from [PK*].)

With these specifications, we see that $(u, w, q) \in \Gamma(v)$, and is a solution to the original problem defined by $\mathsf{T}Q$. So for any $Q \in \mathscr{F}_2$,

$$\mathsf{T}Q(v) = \int_0^1 \big[\max[1-c, 1-d] + \delta v^* + \delta Q(v^*)\big] f(c)\,\mathrm{d}c - v$$

for all $v \geq v^*$. Indeed, with this contract, for any $v, v' \geq v^*$, $\mathsf{T}Q(v) - \mathsf{T}Q(v') = -(v - v')$, that is, $(\mathsf{T}Q)'(v) = -1$ for all $v \geq v^*$. This proves that $\mathsf{T}Q \in \mathscr{F}_2$.

We have established above that by [PK] and [IC], $u(1)+\delta w(1) = v - \int_0^1 q(c)F(c)\,\mathrm{d}c$. Because $u(1), w(1) \geq 0$ (by feasibility) and because $q$ is a monotone function, it follows that $0 \leq \int_0^1 q(c)F(c)\,\mathrm{d}c \leq v$. Therefore, as $v \searrow 0$, $q \searrow 0$ almost surely, and in particular, in $L^1$. Therefore, for any $Q \in \mathscr{F}_3$,

$$\mathsf{T}Q(v) \leq \sup_{(u,w,q)\in\Gamma(v)} \int_0^1 \big[(1-d) + (d-c)q(c)\big] f(c)\,\mathrm{d}c + \sup_{(u,w,q)\in\Gamma(v)} \delta \int_0^1 \big[w(c) + Q\big(w(c)\big)\big] f(c)\,\mathrm{d}c - v$$

The first problem is essentially a static optimisation problem, and so has the solution that $q(c) = 1$ for $c \leq c^*(v)$, while $q(c) = 0$ otherwise. It is easy to see that $c^*(v) \searrow 0$ as $v \searrow 0$. Therefore, the first term tends to 0. Similarly, for the second term, we must have $\delta w(1) = v - \int_0^1 q(c)F(c)\,\mathrm{d}c$, and $\delta w(c) = \delta w(1) + \int_c^1 q(x)\,\mathrm{d}x$, and because

$Q$ is continuous at 0, it follows that the second term converges to $(1-d)/(1-\delta)$. This proves that $\mathsf{T}Q$ is also continuous at 0.

It is easy to see that $\mathsf{T}$ is monotone ($Q_1 \leq Q_2$ implies $\mathsf{T}Q_1 \leq \mathsf{T}Q_2$) and satisfies discounting ($\mathsf{T}(Q+a) = \mathsf{T}Q + \delta a$ where $0 < \delta < 1$) which implies that $\mathsf{T}$ is a contraction mapping on $\mathscr{F}$. We have just established above that if $Q \in \mathscr{F}_1 \cap \mathscr{F}_2 \cap \mathscr{F}_3$, then $\mathsf{T}Q \in \mathscr{F}_1 \cap \mathscr{F}_2 \cap \mathscr{F}_3$. But this implies that the unique fixed point of $\mathsf{T}$, which we shall call $P$, also lies in $\mathscr{F}_1 \cap \mathscr{F}_2 \cap \mathscr{F}_3$ — see Stokey, Lucas and Prescott (1989, Corollary 1, p 52). Finally, observe that $P$ is concave on $[0,\infty)$, and because every concave function is continuous in the interior of its domain, $P$ is continuous everywhere.

Let us define

$$\begin{aligned}\Gamma_0(v) := \{w \in \mathbf{C}[0,1] : w \geq 0, \text{has } \delta \textstyle\int_0^1 w(c) f(c)\,\mathrm{d}c = v, \text{is convex,}\\ \text{decreasing, and has } \partial w(c) \subset [-1,0]\ \forall\, c \in [0,1]\}\end{aligned}$$

where $\partial w(c)$ is the subdifferential of the convex function $w$. Now define the map $\Psi : \Gamma_0(v) \to \Gamma(v)$ defined as $\Psi(w) = (0, w, q)$ where $-q(c) \in \partial w(c)$ for all $c \in [0,1]$, and notice that $\Psi$ is an injective map. Then, for any $w \in \Gamma_0(v)$, $\Psi w$ satisfies [PK] by definition of $\Gamma_0(v)$, [IC] because $-q(c) \in \partial w(c)$, [Mon] because $w$ is continuous and convex which implies $q$ is monotone decreasing, and Feasibility because $\partial w \subset [0,1]$ which implies $q \in [0,1]$ and $w \geq 0$ which ensures liquidity and participation constraints hold.

It is easy to see, most notably because $u$ doesn't appear on the right hand side of [VF-sup], that in any contract, we may set $u = 0$ and let all the utility due the agent accrue via $w$. Therefore, it is without loss of generality to restrict attention to the choice set $\Gamma_0(v)$ for all $v \geq 0$, when considering the supremum in [VF-sup]. Note that for each $v \geq 0$, $\Gamma_0(v)$ is closed and bounded as a subset of $\mathbf{C}[0,1]$. But all the functions in $\Gamma_0(v)$ have the same Lipschitz constant, namely 1, which means that $\Gamma_0(v)$ is equicontinuous. Therefore, $\Gamma_0(v)$ is compact by the Arzela-Ascoli Theorem – see, for instace, Ok (2007).

Because $P(v) + v \in \mathbf{C}_b[0,\infty)$, it follows that the supremum on the right hand side of [VF-sup] is reached for every $v \geq 0$. Moreover, by Berge's Theorem of the Maximum – see Ok (2007) – the optimal policy $(u, w, q)(v)$ is continuous in $v$.

To see part (b), we will use the characterisation of the optimal allocations provided in section 4.2. Suppose $\hat{v} = \min\{v : P'(v) = -1\} < v^*$, so that $P'(\hat{v}) = P'(v^*) = -1$. Then, at $\hat{v}$, $\eta(\hat{v}) = 0$ (by the envelope condition), so by [4.3], we must necessarily have $\int_0^1 \big[P'(w(c)) + 1\big] f(c)\,\mathrm{d}c = 0$, which implies $P'(w(c)) = -1$ for almost all $c$. But $w$ is monotone decreasing and continuous, so we must have $P'(w(c)) = -1$ for all $c \in [0,1]$.

Substituting $\int_0^1 \big[P'(w(c)) + 1\big] f(c)\,\mathrm{d}c = 0$ into [Adj-w], we find that $\lambda(c) = 0$ for all $c$. This, in turn, implies that in [4.4], $\xi(c)$ is as in a static problem, and allocations are first best. That is, $q(c, \hat{v}) = 1$ if $c \le d$, while $q(c, \hat{v}) = 0$ if $c > d$.

As noted before, we may restrict attention, without loss of generality, to contracts where $u = 0$. Then, $\delta w(1) = \hat{v} - \int_0^d F(c)\,\mathrm{d}c = \hat{v} - (1-\delta)v^*$. Therefore, $\delta\big(w(1) - \hat{v}\big) = (1-\delta)(\hat{v} - v^*) < 0$, which implies $w(1) < \hat{v}$, because $\hat{v} < v^*$. But we have already established that $P'(w(1)) = -1$, which is a contradiction.

We now establish part (c) and show that $P'(0) = \infty$. It is easy to see that $P(0) = (1-d)/(1-\delta)$. Since $P$ is concave, we know $P'(0) \ge \big[P(v) - P(0)\big]/v$ for all $v > 0$. Recall that $P_0(v)$ is the value function associated with the static optimisation problem (see appendix A). Therefore, for each $v \ge 0$, $P(v) \ge P_0(v)/(1-\delta)$, while $P(0) = P_0(0)/(1-\delta)$. This gives us the bound

$$\begin{aligned} P'(0) &= \lim_{v \searrow 0} \frac{P(v) - P(0)}{v} \\ &\ge \lim_{v \searrow 0} \frac{P_0(v) - P_0(0)}{v} \frac{1}{1-\delta} \\ &= P_0'(0) = \infty \end{aligned}$$

where we have used the fact that $P_0'(0) = \infty$, which was established in lemma A.1. □

## C. Proofs from Sections 4

*Proof of Proposition 4.1.* For a fixed $v$, it is easy to see that the static part of the virtual cost function, $c + \eta F(c)/f(c)$ is continuous in $c$. It is easy to see that $\int_0^c \big[P'(w(s)) + 1\big] f(s)\,\mathrm{d}s$ is continuous in $c$, and because $f(c) > 0$, it follows that the dynamic part of the virtual cost is also continuous in $c$. Therefore, $\psi(c, v)$ is continuous in $c$, proving the first claim.

To prove the second claim, let us define $X(s) := P'(w(s)) + 1$, so that from equation [4.1], $\eta = \int_0^1 X(s) f(s)\,\mathrm{d}s$. Given that $\psi(c, v)$ is continuous in $c$, it suffices to show that $\psi(d, v) > d$. This is equivalent to showing that $\eta F(d) > \delta \int_0^d X(s) f(s)\,\mathrm{d}s$. We will prove the claim for $\delta = 1$, from which the claim will follow for all $\delta \in [0, 1)$.

Notice that $\eta F(d) = F(d)\big[\int_0^d X(s) f(s)\,\mathrm{d}s + \int_d^1 X(s) f(s)\,\mathrm{d}(s)\big]$, so that $\eta F(d) - \int_0^d X(s) f(s)\,\mathrm{d}s = F(d) \int_d^1 X(s) f(s)\,\mathrm{d}s - (1 - F(d)) \int_0^d X(s) f(s)\,\mathrm{d}s$. Recall that $w(s)$ is constant on $[d, 1]$, so $\int_d^1 X(s) f(s)\,\mathrm{d}s = X(d)[1 - F(d)]$. Moreover, $w(s)$ is decreasing in $s$, but because $P$ is concave, $P'(\cdot)$ is a decreasing function, which implies $P'(w(d)) \ge P'(w(s))$ for all $s \in [0, d]$. Therefore, $\int_0^d X(s) f(s)\,\mathrm{d}(s) < X(d) F(d)$, where the inequality is strict because $w(s)$ is not constant on $[0, d]$ for any $v < v^*$.

Putting these facts together, we see that

$$\begin{aligned}
&\eta F(d) - \int_0^d X(s) f(s)\,\mathrm{d}s \\
&\quad = F(d) \int_d^1 X(s) f(s)\,\mathrm{d}s - (1 - F(d)) \int_0^d X(s) f(s)\,\mathrm{d}s \\
&\quad = F(d) X(d)[1 - F(d)] - (1 - F(d)) \int_0^d X(s) f(s)\,\mathrm{d}s \\
&\quad = [1 - F(d)] \left[ X(d) F(d) - \int_0^d X(s) f(s)\,\mathrm{d}s \right] \\
&\quad > 0
\end{aligned}$$

which completes the proof. □

*Proof of Proposition 4.2.* Suppose, by way of contradiction, that bang-bang allocations are optimal. Fix $v \geq 0$ (which we shall take to be sufficiently small), and suppose for this $v$, $q(c, v)$ is bang-bang. Then, there exists $x$ (which depends on $v$ such that for all $c \leq x$, $q(c, v) = 1$, while for all $c > x$, $q(c, v) = 0$. By [PK*], this allocation induces continuation utility $\delta w(1) = v - \int_0^1 q(c) F(c)\,\mathrm{d}c$. Moreover, $\delta w(c) = \delta w(1) + \int_c^1 \mathrm{d}t = \delta w(1) + \max[x - c, 0]$.

Fix $\varepsilon > 0$ to be small, such that $x - \varepsilon > 0$. Now consider an allocation $\tilde{q}$ such that $\tilde{q} = q$ on $[0, x - \varepsilon) \cup (x, 1]$, while $\tilde{q} = 1 - h$ on $[x - \varepsilon, x]$, where $h \in (0, 1)$. With this specification, $\int_0^1 (q - \tilde{q})\,\mathrm{d}c = h\varepsilon > 0$.

Let $\tilde{w}$ represent the induced continuation utility. Then, $\delta\tilde{w}(1) - \delta w(1) = \int_0^1 (q - \tilde{q}) F\,\mathrm{d}c = \frac{h\varepsilon}{2}[2x - \varepsilon] > 0$, while $\delta\tilde{w}(1) - \delta w(1) = \delta\big[\tilde{w}(0) - w(0)\big] - \int_0^1 (q - \tilde{q})\,\mathrm{d}c = h\varepsilon[x - \varepsilon/2 - 1] < 0$.

The difference in instantaneous utility from choosing $q$ instead of $\tilde{q}$ is $\int_0^1 (d - c)(q - \tilde{q})\,\mathrm{d}c = \int_{x-\varepsilon}^x (d - c) h\,\mathrm{d}c = h\varepsilon[d - x + \varepsilon/2]$.

Notice that $\tilde{w}(0) < w(0)$, so some types lose continuation utility by switching to $\tilde{w}$. But all these types have cost less than $x$, since for all types $c \geq x$, $\tilde{w}(c) = \tilde{w}(1) > w(1) = w(c)$. Therefore, the loss in continuation utility (in switching from $w$ to $\tilde{w}$) is no greater than $0 \leq \delta \int_0^x \big[P(w) - P(\tilde{w}) + w - \tilde{w}\big]\,\mathrm{d}c$. Notice that in this range, $P(w) \geq P(\tilde{w})$, and for all $c$, $P(w(c)) - P(\tilde{w}(c)) \leq P'(\tilde{w}(c))(w(c) - \tilde{w}(c)) \leq P'(\tilde{w}(1))(w(0) - \tilde{w}(0))$. Therefore,

$$\begin{aligned}
&\delta \int_0^x \big[P(w) - P(\tilde{w}) + w - \tilde{w}\big]\,\mathrm{d}c \\
&\quad \leq \delta \int_0^x \big[P'(\tilde{w}(1))(w(0) - \tilde{w}(0)) + w(0) - \tilde{w}(0)\big]\,\mathrm{d}c \\
&\quad \leq \delta h\varepsilon(1 + \varepsilon/2 - x)\big[P'(\tilde{w}(1)) + 1\big] x
\end{aligned}$$

By moving to $\tilde{w}$ from $w$, all types with cost greater than $x$ gain in continuation utility. This gain is $\delta \int_x^1 \left[P(\tilde{w}) - P(w) + \tilde{w} - w\right] \mathrm{d}c$. Because $P(\tilde{w}(1)) - P(w(1)) \geq P'(\tilde{w}(1))\big(\tilde{w}(1) - w(1)\big)$, the gain in continuation utility is at most

$$\delta h\varepsilon(2x - \varepsilon)\big[P'(\tilde{w}(1)) + 1\big](1 - x)$$

Therefore, the change to $(\tilde{q}, \tilde{w})$ from $(q, w)$ is profitable if the gain in utility is greater than the loss in utility. The difference between gain and loss is, modulo a factor of $h\varepsilon$ that multiplies all terms,

$$\delta\big[P'(\tilde{w}(1)) + 1\big]\big[2x - \varepsilon - x(1 + x - \varepsilon/2)\big] - [d - x + \varepsilon/2]$$

If $v$ is sufficiently small, then $x$ must also be sufficiently small, else $\delta w(1) = v - \int_0^1 q(c)F(c)\,\mathrm{d}c \geq 0$ would be violated. It then follows that if we take $\varepsilon$ sufficiently small, we can ensure that $2x - \varepsilon - x(1 + x - \varepsilon/2) > 0$. But because $P'(\tilde{w}(1))$ is very large when $\varepsilon$, $v$, and $x$ are very small, we see that the gain outweighs the loss, so it is profitable to switch to the contract $(\tilde{u}, \tilde{w}, \tilde{q})$. This establishes that the optimal allocation policy cannot be bang-bang. □

## D. Proofs from Sections 5

*Proof of proposition 5.2.* Recall that $w(c, v)$ is decreasing in $c$.. The claim is that for all $v \in (0, v^*)$, $P'(w(1, v)) > P'(v) > P'(w(0, v))$. So suppose the claim is not true. Since $P'$ is a martingale, the only possibility then is that $P'(w(0, v)) = P'(w(1, v)) = P'(v)$.

Recall that $\delta w(0, v) - \delta w(1, v) = \int_0^1 q(c)\,\mathrm{d}c$. We now proceed to show, by contradiction, that $P'(w(0, v)) = P'(w(1, v)) = P'(v)$ is impossible. Let $v_0 := v$, so that $w(1, v_0) < v_0 < w(0, v_0)$ and $P'(v) > -1$. Consider the sequence $v_k := w(0, v_{k-1})$, and suppose, as the induction hypothesis, that $P'(\cdot)$ is constant (and strictly greater than $-1$) on the interval $[w(1, v_0), v_k]$, with $v_{k-1} \in [w(1, v_0), v_k]$.

Since a positive fraction of types always produce, it follows that $v_k = w(0, v_{k-1}) > v_{k-1}$which in turn implies that $P'(v_k) = P'(v_{k+1})$. Therefore, $P'(\cdot)$ is constant (and strictly greater than $-1$) on the interval $[w(0, v_0), v_{k+1}]$. Since $(v_k)$ is a strictly increasing sequence that diverges to infinity, we see that $P'(\cdot)$ must then be constant and strictly greater than $-1$ on the interval $[w(0, v_0), \infty)$, which is impossible because $P'(v) = -1$ for $v \geq v^*$. This completes the proof. □

We now move to the proof of Theorem 3. Once again, we follow Thomas and Worrall (1990).

*Proof of Theorem 3.* Since $P'$ is a martingale that is bounded below by $-1$, it follows that $P' + 1$ is a nonnegative martingale. The Martingale Convergence Theorem (see, for instance, Theorem 9.4.4 on p 350 of Chung (2001) and its corollary on p 351), says that $P' + 1$ converges almost surely to a nonnegative, integrable limit, $P'_\infty + 1$. Therefore, $P'$ converges almost surely to $P'_\infty$, and the limit is integrable (which implies that $P'_\infty = \infty$ with zero probability). We want to show that $P'_\infty = -1$ almost surely.

Before getting to the details, it is useful to sketch the intuition. Consider a sample path $(c^{(t)})$ such that $P'$ converges along this sample path. Suppose that along this sample path, $P'$ converges to some number $C > -1$ and $\hat{v}$ is such that $P'(\hat{v}) = C$. It must be that eventually, all the values that $P'$ takes in this sample path must lie arbitrarily close to $C$. Therefore, along this path, the step size of the continuation promises $w(0, c^{(t)}) - w(1, c^{(t)})$ must converge to zero. But this would violate proposition 5.2, which says that $w_1(\hat{v}) - w_2(\hat{v})$ is bounded away from zero and the fact that the optimal contract is continuous in $v$ (which follows easily from the fact that $v$ enters as a parameter in the Hamiltonian).

By proposition 5.2, it follows that there exists a function $c_\circ(v)$ such that for all $c \geq c_\circ(v)$, $P'(w(c_\circ(v), v)) > P'(v)$. Furthermore, it easy to show that this function can be taken to be continuous. In light of this fact, in any neighbourhood of $v$, we can take $c_\circ(v)$ to be independent of $v$. We shall denote this uniform function as $c_\circ$ in this specified neighbourhood.

Consider a sample path with the properties that (i) $\lim_{n\to\infty} P'(v^n) = C \notin \{-1, \infty\}$, and (ii) state $\{c : c \geq c_\circ\}$ occurs infinitely often. Define $C =: P'(y)$, so that $\lim_{n\to\infty} v^n = y$ and consider a subsequence $(\sigma(n))$ such that $c^{\sigma(n)} \geq c_\circ$ for all $n$, ie this is the subsequence consisting of all the cost shocks in the original sequence that lie in the set $\{c : c \geq c_\circ\}$. Since $(v^{\sigma(n)})$ is a subsequence of $(v^n)$, it also converges to $y$.

Recall that the evolution of promised utility along any sample path can be written as $\varphi(v^n, c^{(n)}) = v^{n+1}$, where $\varphi(v, \cdot)$ is continuous in $v$. This induces the function $\varphi^\sigma(v, c^{(n)})$ where $\varphi^\sigma(v^{\sigma(n)}, c^{(n)}) = v^{\sigma(n+1)}$. Since $\varphi(v, \cdot)$ is continuous in $v$, it follows that $\varphi^\sigma(v, \cdot)$ is also continuous in $v$. Therefore, the sequence $\varphi^\sigma(v^{\sigma(n)}, c^{(n)})$ converges to $\varphi^\sigma(y, c)$. Moreover, $\varphi^\sigma(y, c) = \varphi(y, c) = y$ for all $c \geq c_\circ$, since $\varphi^\sigma(v^{\sigma(n)}, c) = v^{\sigma(n+1)}$, and $\lim_{n\to\infty} v^{\sigma(n)} = \lim_{n\to\infty} v^n = y$.

But $\lim_{n\to\infty} P'(v^{\sigma(n)}) = C$ and $\lim_{n\to\infty} P'(v^{\sigma(n+1)}) = C$, so by the continuity of $P'$ we have $P'(y) = P'\big(\varphi^\sigma(y, c)\big) = P'\big(\varphi(y, c)\big) = C$ for all $c \geq c_\circ$, contradicting proposition 5.2 which states that $P'(y) < P'\big(\varphi(y, c)\big)$ for all $c \geq c_\circ$. In light of the fact that paths where the set $\{c : c \geq c_\circ\}$ does not occur infinitely often are of probability zero, we have proved the proposition. □

## References


Bergemann, Dirk and Juuso Välimäki (2010). 'The Dynamic Pivot Mechanism'. In: *Econometrica* 78.2, pp. 771–789 (cit. on p. 6).

Biais, Bruno et al. (2007). 'Dynamic Security Design: Convergence to Continuous Time and Asset Pricing Implications'. In: *Review of Economic Studies* 74.2, pp. 345–390 (cit. on p. 5).

Boleslavsky, Raphael and Maher Said (2012). 'Progressive Screening: Long Term Contracting With a Privately Known Stochastic Process'. In: *Review of Economic Studies* forthcoming (cit. on p. 6).

Chung, Kai Lai (2001). *A Course in Probability Theory*. 3rd. Academic Press (cit. on p. 28).

Clarke, Frank H (1983). *Optimization and Nonsmooth Analysis*. New York: John Wiley and Sons (cit. on p. 14).

Clementi, Gian Luca and Hugo A. Hopenhayn (2006). 'A Theory of Financing Constraints and Firm Dynamics'. In: *Quarterly Journal of Economics* 121, pp. 229–265 (cit. on p. 5).

Cousins, Paul D. et al. (2006). 'Creating Supply Chain Relational Capital: The impact of formal and informal socialization processes'. In: *Journal of Operations Management* 24.6, pp. 851–863 (cit. on p. 2).

DeMarzo, Peter and Michael J. Fishman (2007). 'Agency and Optimal Investment Dynamics'. In: *Review of Financial Studies* 20.1, pp. 151–188 (cit. on p. 5).

DeMarzo, Peter and Yuliy Sannikov (2006). 'Optimal Security Design and Dynamic Capital Structure in a Continuous-Time Agency Model'. In: *Journal of Finance* 61.6, pp. 2681–2724 (cit. on p. 5).

Green, Edward J. (1987). 'Lending and the Smoothing of Uninsurable Income'. In: *Contractual Agreements for Intertemporal Trade*. Ed. by Edward C. Prescott and Neil Wallace. University of Minnesota Press (cit. on p. 5).

Guesnerie, Roger and Jean-Jacques Laffont (1984). 'A Complete Solution to a Class of Principal-Agent Problems with an Application to the Control of a Self-Managed Firm'. In: *Journal of Public Economics* 25, pp. 329–369 (cit. on p. 14).

Hellwig, Martin (2009). *A Maximum Principle for Control Problems with Monotonicity Constraints*. Tech. rep. Bonn: Max Planck Institute (cit. on pp. 6, 14, 15).

Krause, Daniel R., Robert B. Handfield and Beverly B. Tyler (2007). 'The relationships between supplier development, commitment, social capital accumulation and performance improvement'. In: *Journal of Operations Management* 25.2, pp. 528–545 (cit. on p. 2).

Krishna, R. Vijay, Giuseppe Lopomo and Curtis R. Taylor (2013). 'Stairway to Heaven or Highway to Hell: Liquidity, Sweat Equity, and the Uncertain Path to Ownership'. In: *RAND Journal of Economics* 44.1, pp. 104–127 (cit. on pp. 6, 17, 21).

Laffont, Jean-Jacques and David Martimort (2002). *The Theory of Incentives*. Princeton, NJ: Princeton University Press (cit. on p. 6).

Milgrom, Paul (1987). *Adverse Selection without Hidden Information*. Tech. rep. 8742. University of California, Berkeley (cit. on p. 5).

Myerson, Roger B. (1981). 'Optimal Auction Design'. In: *Mathematics of Operations Research* 6.1, pp. 58–73 (cit. on pp. 6, 8).

Ok, Efe A. (2007). *Real Analysis with Economic Applications*. Princeton University Press (cit. on p. 24).

Pavan, Alessandro, Ilya Segal and Juuso Toikka (2014). 'Dynamic Mechanism Design: A Myersonian Approach'. In: *Econometrica* 82.2, pp. 601–653 (cit. on pp. 5, 6).

Quadrini, Vincenzo (2004). 'Investment and liquidation in renegotiation-proof contracts with moral hazard'. In: *Journal of Monetary Economics* 51.4, pp. 713–751 (cit. on p. 5).

Spear, Stephen E and Sanjay Srivastava (1987). 'On Repeated Moral Hazard with Discounting'. In: *Review of Economic Studies* 54.4, pp. 599–617 (cit. on p. 5).

Stokey, Nancy L., Robert E. Lucas Jr and Edward C. Prescott (1989). *Recursive Methods in Economic Dynamics*. Cambridge, Ma: Harvard University Press (cit. on pp. 23, 24).

Takayama, Akira (1985). *Mathematical Economics*. Cambridge, UK: Cambridge University Press (cit. on p. 12).

Thomas, Jonathan P. and Tim Worrall (1990). 'Income Fluctuation and Asymmetric Information: An Example of a Repeated Principal-Agent Problem'. In: *Journal of Economic Theory* 51, pp. 367–390 (cit. on pp. 4, 5, 8, 27).